Portfolio Analysis: A Demonstration

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Solution

a) We have the following utility function $U(r_p) = -\exp\{-\theta r_p\}$. It is required to maximize the expectation of the utility function

$$\mathbb{E}\left(-\exp\{-\theta r_p\}\right) \to \max \quad \text{subject to } \mathbb{1}_N' x = 1.$$

Therefore, the expectation of the utility function is

$$-\exp\{-\theta\,\mathbb{E} r_p + \frac{\theta^2}{2}\operatorname{Var} r_p\} = -\exp\{-\theta x'\mu + \frac{\theta^2}{2} x'\Sigma\, x\} \to \quad \max$$

which is equivalent to

$$\arg\max_x \left(\theta x'\mu - \frac{\theta^2}{2} x'\Sigma\, x\right) \quad \text{subject to} \quad \mathbb{1}_N' x = 1. \tag{1}$$

It is known that $\frac{\partial a'x}{\partial x} = a$, $\frac{\partial x'a}{\partial x} = a$, $\frac{\partial x'\Sigma x}{\partial x} = (\Sigma + \Sigma')x = 2\Sigma\, x$ and the Lagrangian function for problem (1) can be written as

$$\mathcal{L} = \theta x'\mu - \frac{\theta^2}{2} x'\Sigma\, x + \lambda(1 - \mathbb{1}_N' x) \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \theta\mu - \theta^2 \Sigma\, x - \lambda \mathbb{1}_N = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \mathbb{1}_N' x = 0. \tag{4}$$

Solving (3) for x it follows that

$$x = \frac{1}{\theta^2}\Sigma^{-1}(\theta\mu - \lambda \mathbb{1}_N) \tag{5}$$

and plugging (5) in (4) we have

$$
\begin{aligned}
1 - \frac{1}{\theta^2}\mathbb{1}_N'\Sigma^{-1}(\theta\mu - \lambda\mathbb{1}_N) &= 0 \\
\mathbb{1}_N'\Sigma^{-1}(\theta\mu - \lambda\mathbb{1}_N) &= \theta^2 \\
\mathbb{1}_N'\Sigma^{-1}\mathbb{1}_N\lambda &= \theta\mathbb{1}_N'\Sigma^{-1}\mu - \theta^2 \\
\lambda &= \frac{\theta\mathbb{1}_N'\Sigma^{-1}\mu - \theta^2}{\mathbb{1}_N'\Sigma^{-1}\mathbb{1}_N}.
\end{aligned} \tag{6}
$$

Now substitute (6) in (5) and we obtain

$$
\begin{aligned}
x &= \frac{1}{\theta^2}\Sigma^{-1}\left(\theta\mu - \frac{\theta\mathbb{1}_N'\Sigma^{-1}\mu}{\mathbb{1}_N'\Sigma^{-1}\mathbb{1}_N}\mathbb{1}_N + \frac{\theta^2}{\mathbb{1}_N'\Sigma^{-1}\mathbb{1}_N}\mathbb{1}_N\right) \\
&= \frac{\Sigma^{-1}\mathbb{1}_N}{\mathbb{1}_N'\Sigma^{-1}\mathbb{1}_N} + \frac{1}{\theta}\Sigma^{-1}\left(\mu - \frac{\mathbb{1}_N\Sigma^{-1}\mu}{\mathbb{1}_N'\Sigma^{-1}\mathbb{1}_N}\mathbb{1}_N\right) \\
&= \frac{\Sigma^{-1}\mathbb{1}_N}{c} + \frac{1}{\theta}\Sigma^{-1}\left(\mu - \frac{b}{c}\mathbb{1}_N\right) \\
&= x_{GMVP} + \frac{1}{\theta}\Sigma^{-1}(\mu - \mu_{GMVP}\mathbb{1}_N).
\end{aligned} \tag{7}
$$

For $N = 3$ the proof is completed.

b) We have weekly (Thursday to Thursday) percentage returns of the S&P500, FTSE, and DAX indices over the period from January 1984 to August 2005, a sample of 1127 observations. The return indices are plotted in Figure 1.
The sample moments are

$$
\hat{\mu} = \frac{1}{500}\sum_{t=1}^{500} r_t, \quad \widehat{\Sigma} = \frac{1}{499}\sum_{t=1}^{500}(r_t - \hat{\mu})(t_t - \hat{\mu})' \tag{8}
$$

(i) The mean sample vector $\widehat{\mu}$ and sample covariance matrix are

$$
\widehat{\mu} = \begin{pmatrix} \bar{r}_{\text{S\&P500}} \\ \bar{r}_{\text{FTSE}} \\ \bar{r}_{\text{DAX}} \end{pmatrix} = \begin{pmatrix} 0.217 \\ 0.263 \\ 0.309 \end{pmatrix}, \quad \widehat{\Sigma} = \begin{pmatrix} 4.35 & 2.58 & 2.05 \\ 2.58 & 7.70 & 3.86 \\ 2.05 & 3.86 & 8.47 \end{pmatrix}. \tag{9}
$$

Hence, the correlation matrix is

	sp500	ftse	dax
sp500	1.00	0.45	0.34
ftse	0.45	1.00	0.48
dax	0.34	0.48	1.00

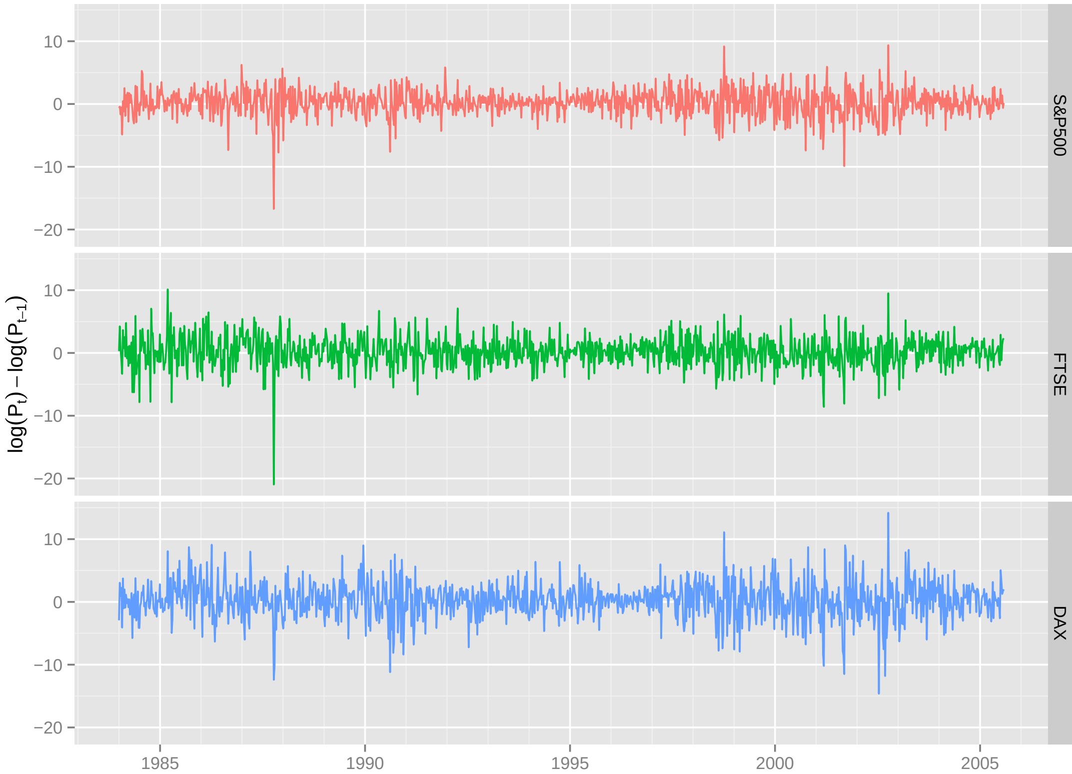

Figure 1: Weekly percentage returns of S&P500, FTSE, and DAX over the period from January 1984 to August 2005.

As expected, there is a weak positive correlation between all stock market indices. Therefore, an increase of one stock index affects the behaviour of all other market indices in the same way.

(ii) To compute the optimal potfolio weights for both investors we use (7)

$$x_i^\star = x_{\text{GMVP}} + \frac{1}{\theta_i}\Sigma^{-1}(\hat{\mu} - \mu_{\text{GMVP}}\mathbb{1}_3), \quad \text{for } i = 1, 2. \tag{10}$$

Using $\hat{\mu}$ and $\widehat{\Sigma}$ as given in (9), we have

$$\begin{aligned}
&a = 0.0173,\ b = 0.0672,\ c = 0.2765, \text{ and } d = 0.00027\\
&x_{\text{GMVP}} = \frac{1}{\mathbf{1}_3'\Sigma^{-1}\mathbf{1}_3}\Sigma^{-1}\mathbf{1}_3 = \frac{1}{c}\Sigma^{-1}\mathbf{1}_3 = (0.65, 0.15, 0.20)'\\
&\mu_{\text{GMVP}} = x_{\text{GMVP}}'\hat{\mu} = \frac{b}{c} = 0.243.
\end{aligned}$$

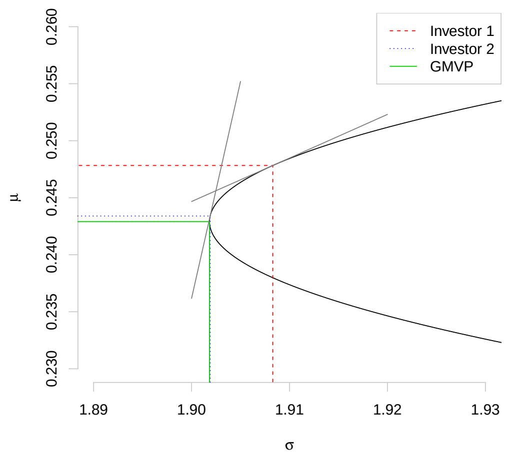

Figure 2: The minimum variance set (MVS) as implied by the estimates (9) for both investors.

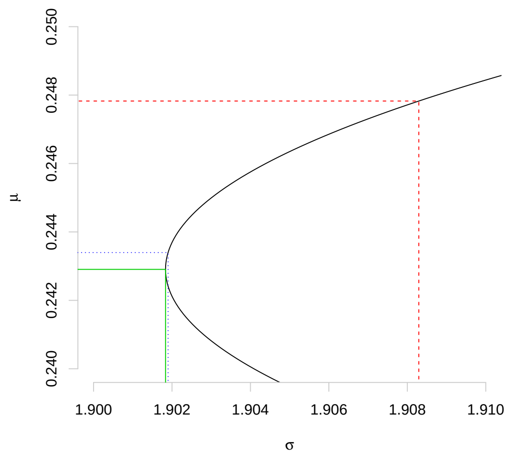

Figure 3: A magnified version of the figure to the left.

Therefore, the optimal portfolio weights for $\theta_1 = 0.2$ and $\theta_2 = 2$ are

$$x_1^\star = (0.59, 0.16, 0.25)' \quad \text{and}$$
$$x_2^\star = (0.64, 0.15, 0.21)'.$$

To plot the graph of the minimum variance set in $(\sigma_p - \mu_p)$ space we use

$$\sigma_{\text{MVS}} = \sqrt{\frac{a - 2b\mu_i^* + c\mu_i^{*2}}{d}}. \tag{11}$$

Therefore, the "coordinates" in the mean–variance space for both investors are

	μ_p	σ_p
Investor 1	0.247	1.9082
Investor 2	0.243	1.9019
GMVP	0.242	1.9018

Table 1: Mean and standard deviation of the investors' portfolios.

The minimum variance set (MVS) from Equation (11) is depicted in Figure 2. A closer look, shown in Figure (3), emphasizes the slight difference between σ_p of Investor 2 and the σ_{GMVP}.

(iii) The two investors use the estimated weights $x_1^\star = (0.59, 0.16, 0.25)$ and $x_2^\star = (0.64, 0.15, 0.21)$ for the remaining weeks. The *'new'* portfolio returns are calculated via

$$r_{p,t}^1 = x_1^{\star\prime} r_t, \quad t = 501, \ldots, 1127$$

and the means are

$$\bar{r}_p^1 = \frac{1}{627} \sum_{t=501}^{1127} r_{p,t}^1 = 0.187 \quad \text{and} \quad \bar{r}_p^2 = \frac{1}{627} \sum_{t=501}^{1127} r_{p,t}^2 = 0.186 \tag{12}$$

respectively. We plug (12) into (11) and get the following out-of-sample estimations:

	μ_p	σ_p
Investor 1	0.187	2.590
Investor 2	0.186	2.615

Table 2: Out-of-sample mean and standard deviation of the investors' portfolios.

We have considerably lower means and higher standard deviations since we do not recalculate the optimal weights with the new information. Since wealth changes, we should update the weights, for example, on a weekly basis. Furthermore, the stock markets seem to have become more integrated in recent years which additionally increases the portfolio variance. The out-of-sample correlation matrix is

	sp500	ftse	dax
sp500	1.00	0.65	0.66
ftse	0.65	1.00	0.72
dax	0.66	0.72	1.00