Portfolio Analysis WS12/13: Computer Excercise

Problem 1

Consider a simple problem of international portfolio diversification. An investor wants to allocate wealth between the major stock market indices of the US, the United Kingdom and Germany. We consider all–equity allocations, i.e., we assume that there is no risk–free asset.

The file `Rweek` (Matlab or Excel) contains weekly (Thursday to Thursday) percentage returns of the S&P500, FTSE, and DAX indices over the period from January 1984 to August 2005, a sample of 1127 observations. The returns are denominated in terms of dollars, i.e., we assume the perspective of an US–investor not hedging currency risk. We denote the return vector at time t by $\boldsymbol{r}_t = [r_{1t}, r_{2t}, r_{3t}]'$, where r_{1t}, r_{2t}, and r_{3t} are the time–t returns of the S&P500, the FTSE, and the DAX, respectively.

Assume that the returns under study have a multivariate normal distribution with mean $\boldsymbol{\mu} = [\mu_1, \mu_2, \mu_3]'$ and covariance matrix $\boldsymbol{\Sigma}$, and that the expected utility function of the investor (with initial wealth 1) is given by

$$U(r_p) = -\exp\{-\theta r_p\}, \quad \theta > 0, \tag{1}$$

where r_p is the portfolio return.

a) Show that, if there are no short–sale restrictions, the optimal vector of portfolio weights, $\boldsymbol{x}^\star$, where $\boldsymbol{x}^\star$ maximizes the expectation of (1) subject to $\mathbf{1}_3'\boldsymbol{x}^\star = 1$, where $\mathbf{1}_3 = [1, 1, 1]'$, is given by

$$\begin{aligned} \boldsymbol{x}^\star &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_3}{\mathbf{1}_3'\boldsymbol{\Sigma}^{-1}\mathbf{1}_3} + \frac{1}{\theta}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \frac{\mathbf{1}_3'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}_3'\boldsymbol{\Sigma}^{-1}\mathbf{1}_3}\mathbf{1}_3\right) \\ &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_3}{c} + \frac{1}{\theta}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \frac{b}{c}\mathbf{1}_3\right) \\ &= \boldsymbol{x}_{GMVP} + \frac{1}{\theta}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \mu_{GMVP}\mathbf{1}_3\right). \end{aligned} \tag{2}$$

b) Now do an out–of–sample portfolio experiment, as follows. Estimate the moments of the return distribution based on roughly the first ten years of data, i.e., the first 500 observations, using the sample moments, i.e.,

$$\widehat{\boldsymbol{\mu}} = \frac{1}{500}\sum_{t=1}^{500} \boldsymbol{r}_t, \quad \widehat{\boldsymbol{\Sigma}} = \frac{1}{499}\sum_{t=1}^{500} (\boldsymbol{r}_t - \widehat{\boldsymbol{\mu}})(\boldsymbol{r}_t - \widehat{\boldsymbol{\mu}})'. \tag{3}$$

(i) Calculate the correlation matrix associated with $\widehat{\boldsymbol{\Sigma}}$ in (3) and briefly describe the correlation structure of the stock market indices. Is it in line with what you would have expected?

(ii) There are two investors, one with $\theta = 0.2$ in (1), and the other with $\theta = 2$ in (1). Find the optimal portfolio weights for both of these (using (2)), denoted by $\boldsymbol{x}_1$ for investor 1, and $\boldsymbol{x}_2$ for investor 2.

Draw the minimum variance set (MVS), as implied by the estimates given in (3), in (σ, μ)–space and mark the optimal (σ, μ)–combinations for both investors.

(iii) Both investors use the optimal weights determined in (ii) to construct portfolios for the remaining 627 weeks.[1] This gives, for both investors, a sequence of 627 *realized* out–of–sample portfolio returns,

$$r_{p,t}^i = \boldsymbol{x}_i' \boldsymbol{r}_t, \quad i = 1, 2, \quad t = 501, \ldots, 1127. \tag{4}$$

Compute the means and the variances of the sequences $\{r_{p,t}^1\}$ and $\{r_{p,t}^2\}$. Compare the results for both investors. Also compare the results with the out–of–sample means and variances of the individual indices, i.e.,

$$\widehat{\boldsymbol{\mu}}_{out} = \frac{1}{627}\sum_{t=501}^{1127} \boldsymbol{r}_t, \quad \widehat{\sigma}_{i,out}^2 = \frac{1}{626}\sum_{t=501}^{1127} (r_{it} - \widehat{\mu}_{i,out})^2, \quad i = 1, 2, 3. \tag{5}$$

(iv) Compute the out–of–sample correlation matrix. Does it seem that stock markets have become more integrated in recent years?

[1] In principle, we would have to recalculate the optimal weights every week, because wealth changes. However, we ignore this, i.e., wealth remains fixed at 1.