Solutions for problem statement 5:

a) **Consider an economy with N risky assets and a risk-free asset. Assume that the following linear relationship (the *excess return market model*) holds between the return on asset i in period t, r_{it}, and the return on the market portfolio in period t, r_{Mt}:**

$$r_{it} - r_{ft} = \alpha_i + \beta_i(r_{Mt} - r_{ft}) + \epsilon_{it}, \quad i = 1, \dots, N, \tag{1}$$

where r_{ft} is the risk-free rate in period t, and

$$E[\epsilon_{it}] = 0, \quad Var[\epsilon_{it}] = \sigma_i^2, \quad t = 1, \dots, T; \quad i = 1, \dots, N, \tag{2}$$

$$Cov[\epsilon_{it}, \epsilon_{js}] = \begin{cases} \sigma_{ij} & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} \quad i, j = 1, \dots, N. \tag{3}$$

Let the covariance matrix of $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \dots, \epsilon_{Nt})'$ be denoted by Σ, i.e.,

$$\Sigma = Cov[\boldsymbol{\epsilon}_t] = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \dots & \sigma_N^2 \end{pmatrix}. \tag{4}$$

The errors are also uncorrelated with the market return at any period, i.e.,

$$Cov[\epsilon_{it}, r_{Ms}] = 0, \quad i = 1, \dots, N; \quad s, t = 1, \dots, T. \tag{5}$$

The *excess return market model* given by the equations (1)-(4) is often adopted as a framework for testing the standard (Sharpe-Lintner) version of the CAPM.

i) **Explain how and which implications for the parameters in (1) can be derived from the Sharpe-Lintner CAPM.**

Consider the single-index model

$$r_{it} = \alpha_i + \beta_i r_{Mt} + \epsilon_{it} \quad (2).$$

Taking expectations yields

$$\mu_i = \alpha_i + \beta_i \mu_M \quad (3).$$

Subtracting (3) from (2) yields

$$r_{it} - \mu_i = \beta_i r_{Mt} - \beta_i \mu_M + \epsilon_{it}.$$

Expressing μ_i via the CAPM one obtains

$$\begin{aligned} r_{it} - r_f - (\mu_M + r_f)\beta_i &= \beta_i r_{Mt} - \beta_i \mu_M + \epsilon_{it} \\ \Rightarrow r_{it} - r_f &= \beta_i (r_{Mt} - r_f) + \epsilon_{it} \end{aligned}$$

The above equation equals the excess return market model with $\alpha_i = 0, i = 1, \ldots, N$. α_i denotes the difference between the asset's average excess return and the excess return predicted by the CAPM.

ii) **In practice, the parameters of the market model are unknown and have to be estimated from historical return data. Given the assumptions made above, suggest a method for estimating α_i and β_i ($i = 1, \ldots, N$) in (1), and justify the proposed procedure.**

Define

$$\underbrace{\tilde{r}_i}_{T\times 1} := \begin{pmatrix} r_{i1} - r_{f1} \\ \vdots \\ r_{iT} - r_{fT} \end{pmatrix}$$

$$\underbrace{\theta_i}_{2\times 1} := \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$$

$$\underbrace{X_i}_{T\times 2} := \begin{pmatrix} 1 & \overbrace{\tilde{r}_{M1}}^{:=r_{M1}-r_{f1}} \\ \vdots & \vdots \\ 1 & \underbrace{\tilde{r}_{MT}}_{:=r_{Mt}-r_{ft}} \end{pmatrix}$$

$$\underbrace{\epsilon_i}_{T\times 1} := \begin{pmatrix} \epsilon_{i1} \\ \vdots \\ \epsilon_{iT} \end{pmatrix}$$
$$i = 1, 2, \ldots, N$$

so that we can write the NT observations in one large set of equations

$$\underbrace{\underbrace{\begin{pmatrix} \tilde{r}_1 \\ \vdots \\ \tilde{r}_N \end{pmatrix}}_{=:\tilde{r}}}_{NT\times 1} = \underbrace{\begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_N \end{pmatrix}}_{=:X} \underbrace{\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}}_{=:\theta} + \underbrace{\begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_N \end{pmatrix}}_{\epsilon} \qquad (\star)$$

<u>Covariance structure of ϵ:</u>
Define

$$\tilde{\epsilon}_t := \begin{pmatrix} \epsilon_{1t} \\ \vdots \\ \epsilon_{Nt} \end{pmatrix},$$

i.e., $\tilde{\epsilon}_t$ collects all asset-specific error terms at one moment in time.
According to equation (2) in the problem statement, the asset-specific error terms may be correlated at the same point in time and this correlation is independent of time. Thus, we allow for a non-diagonal covariance matrix Σ of the vector $\boldsymbol{\epsilon}_t$:

$$\Sigma := Cov[\tilde{\epsilon}_t, \tilde{\epsilon}_t] =: \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \ldots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \ldots & \sigma_N^2 \end{pmatrix}.$$

Equation (2) in the problem statement also states that there is no correlation between the errors over time, i.e.,

$$Cov[\epsilon_i, \epsilon_j] = \sigma_{ij} I_T, \quad i, j = 1, \ldots, N, i \neq j.$$

Hence, we obtain

$$\Omega := Cov[\epsilon, \epsilon] = \begin{pmatrix} \overbrace{\sigma_1^2 I_T}^{Cov[\epsilon_1,\epsilon_1]} & \dots & \overbrace{\sigma_{1N} I_T}^{Cov[\epsilon_1,\epsilon_N]} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} I_T & \dots & \sigma_N^2 I_T \end{pmatrix} \overset{1}{=} \Sigma \otimes I_T \neq \sigma^2 I_{NT}$$

That is, the covariance structure of the error in $(\star)$ is not diagonal.

- Instead, some disturbances are heteroscedastic and correlated in the same time period ($Var[\epsilon_{it}] = \sigma_i^2, Cov[\epsilon_{it}, \epsilon_{jt}] \neq 0, i, j = 1, \dots, N$).
- This system of equations is also called seemingly unrelated regression (SUR-system) [2]
- One can show that the ordinary least-squares estimator is still efficient (in the sense of the Gauss-Markov-Theorem) if each block-equation has the same regressors.
- Here

$$X_i = \begin{pmatrix} 1 & \tilde{r}_{M1} \\ \vdots & \vdots \\ 1 & \tilde{r}_{Mt} \end{pmatrix}, \quad i = 1, \dots, N,$$

  i.e., the regressors of all block-regressions is a constant and r_M.
- That is, we can efficiently estimate (α_i, β_i) using ordinary least-squares.

iii) **The market model has been estimated for 25 stocks in the German DAX index, using $T = 244$ weekly observations for each stock. An estimate for Σ in (3) has been calculated as $\hat{\Sigma} = T^{-1} \sum_{t=1}^T \hat{\boldsymbol{\epsilon}}_t \hat{\boldsymbol{\epsilon}}_t'$, where $\hat{\boldsymbol{\epsilon}}_t$ is an appropriate estimate of $\boldsymbol{\epsilon}_t$. For this estimate of Σ, it was found that $\det(\hat{\Sigma}) = 95.25$. Redoing the estimation, but with the restriction that, in (1), $\alpha_1 = \alpha_2 = \ldots =$**

[1] The Kronecker product is defined as

$$\underbrace{A}_{m\times n} \otimes \underbrace{B}_{p\times q} = \underbrace{\begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \vdots & & & \\ a_{m1}B & & & a_{mn}B \end{bmatrix}}_{mp\times nq}$$

.

[2] It is called seemingly uncorrelated regression, because the regressor matrix X is block-diagonal and therefore $\hat{\theta}_i$ and $\hat{\theta}_j$ seem statistically to be unrelated. But $\hat{\theta}_i$ and $\hat{\theta}_j$ are related due to the non-diagonal covariance structure of the error terms.

$\alpha_N = 0$, **the covariance matrix estimate has a determinant of** $\det(\hat{\Sigma}_{\boldsymbol{\alpha}=\mathbf{0}}) = 117.07$.
Use these numbers to conduct a likelihood ratio test (LRT) for the validity of the standard version of the CAPM ($\alpha = 0.05$).

Testing the standard version of the CAPM gives rise to the following hypotheses:

$$H_0 : \alpha_1 = \alpha_2 = \ldots = \alpha_N = 0 \quad \text{vs.} \quad H_1 : \exists i = 1, \ldots, N : \alpha_i \neq 0.$$

To conduct a test, we can use a likelihood ratio test. To this end, each of the two competing models, the null model and the alternative model, is separately fitted to the data and the log-likelihood is recorded. Then, we compare the maximum likelihood values of the models with and without the restrictions being imposed

$$\lambda = \frac{\overbrace{L_0}^{\text{likelihood for the null model}}}{\underbrace{L_1}_{\text{likelihood for the alternative model}}}.$$

Under regularity conditions one can show that

$$LR := -2ln(\lambda) \overset{T\to\infty}{\sim} \chi^2(\underbrace{r}_{\text{number of restrictions under } H_0})$$

For a SUR-system (with normally distributed errors) it holds

$$\lambda = \left(\frac{\det(\hat{\Sigma}_0)}{\det(\hat{\Sigma}_1)}\right)^{-T/2}$$
$$\Rightarrow LR = T[\underbrace{\ln(\det(\hat{\Sigma}_0)) - \ln(\det(\hat{\Sigma}_1))}_{\geq 0}].$$

Answer to the problem:

$$\det(\hat{\Sigma}_1) = 95.25$$
$$\det(\hat{\Sigma}_0) = 117.07$$
$$\Rightarrow LR = 244(\ln(117.07) - \ln(95.25)) = 50.3292$$

The critical value for a significance level of 5% is given by

$$\chi^2_{25,0.95} = 37.65$$
$$\Rightarrow LR > \chi^2_{25,0.95},$$

i.e., we reject the null hypothesis that the CAPM holds.

b) i) **Based on roughly 5 years of weekly data (November 1996-October 2001), i.e., $T = 244$ observations, the following sample statistics were computed for the excess returns (over the risk-free rate) on the BMW shares, $r_{BMW,t}$, and the German stock market index DAX, $r_{DAX,t}$:**

$$\begin{pmatrix} T & \sum_{t=1}^T r_{DAX,t} \\ \sum_{t=1}^T r_{DAX,t} & \sum_{t=1}^T r^2_{DAX,t} \end{pmatrix} = \begin{pmatrix} 244 & 14.905 \\ 14.905 & 3088.952 \end{pmatrix},$$

$$\begin{pmatrix} \sum_{t=1}^T r_{BMW,t} \\ \sum_{t=1}^T r_{BMW,t}\, r_{DAX,t} \end{pmatrix} = \begin{pmatrix} 63.754 \\ 3151.219 \end{pmatrix},$$

$$\sum_{t=1}^T r^2_{BMW,t} = 8476.325.$$

Using ordinary least-squares (OLS) and the DAX as the market index, estimate the market model (1) for the BMW returns and perform a test with type-I error $\alpha = 0.05$ of the hypothesis that the Standard CAPM with risk-free rate holds for the BMW returns.[3]

$$\hat{\theta} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \underbrace{(X'X)^{-1}}_{\substack{\text{inverse of the} \\ \text{1. matrix in the} \\ \text{problem statement}}} \quad \underbrace{X'y}_{\substack{\text{2. matrix in the} \\ \text{problem statement}}}$$

[3]Strictly speaking, the formulation "the CAPM holds for BMW" is nonsense, because the CAPM is an equilibrium model for the capital market and cannot hold just for a single market. However, it has to hold for each asset separately, of course, so tests involving a single asset may not be useless.

$$= \begin{pmatrix} 0.0041 & 0 \\ 0 & 0.0003 \end{pmatrix} \begin{pmatrix} 63.754 \\ 3151.219 \end{pmatrix} = \begin{pmatrix} 0.199 \\ 1.019 \end{pmatrix}$$

For the CAPM it holds:

$$H_0 : \alpha = 0 \Leftrightarrow t = \left| \frac{\hat{\alpha}}{\sqrt{\widehat{\sigma^2_{\hat{\alpha}}}}} \right| < \underbrace{\tau_c}_{\text{critical value}}$$

To compute the test-statistic we need to compute the variance of $\hat{\alpha}$ which is given by

$$\widehat{\sigma^2_{\hat{\alpha}}} = \widehat{\sigma^2_{\epsilon}} \underbrace{(X'X)^{-1}_{11}}_{\text{(1,1) element of } (X'X)^{-1}} .$$

We compute the variance of the error as follows:

$$\begin{aligned} \hat{\epsilon}'\hat{\epsilon} &= y'y - \hat{\theta}'X'y \\ &= \underbrace{\sum_{t=1}^{T} \tilde{r}^2_{BMW,t}}_{8476.325} - \begin{pmatrix} 0.199 \\ 1.019 \end{pmatrix}' \begin{pmatrix} 63.754 \\ 3151.219 \end{pmatrix} = 5253.170 \\ \Rightarrow \widehat{\sigma^2_{\epsilon}} &= \frac{1}{T-k}\hat{\epsilon}'\hat{\epsilon} = \frac{5253.170}{244-2} = 21.707 \\ \Rightarrow \widehat{\sigma^2_{\hat{\alpha}}} &= 21.707 \times \underbrace{0.0041}_{=(X'X)^{-1}_{11}} = 0.089 \end{aligned}$$

Consequently, the test-statistic and critical values are

$$t = \frac{\hat{\alpha}}{\sqrt{\widehat{\sigma^2_{\hat{\alpha}}}}} = \frac{0.199}{\sqrt{0.089}} = 0.667$$

$$\tau_c = t_{1-\alpha/2}(T-k) = t_{0.975}(242) \approx 1.96 \quad \text{since we have many degrees of freedom.}$$

Therefore, our test decision is

$$|t| < t_{0.975}(242),$$

and we can not reject H_0, i.e., we have found no evidence that the CAPM does not hold.

ii) **Over the same period as above, the market model (1) was jointly estimated**

for the (excess) returns on BMW and Schering. The estimated residual covariance matrix obtained from that estimation is

$$\begin{aligned}\hat{\Sigma}_1 &= \begin{pmatrix} \widehat{\sigma}_{2BMW} & \hat{\sigma}_{BMW,Schering} \\ \hat{\sigma}_{BMW,Schering} & \widehat{\sigma^2}_{Schering} \end{pmatrix} \\ &= \frac{1}{T}\begin{pmatrix} \hat{\boldsymbol{\epsilon}}'_{BMW}\hat{\boldsymbol{\epsilon}}_{BMW} & \hat{\boldsymbol{\epsilon}}'_{BMW}\hat{\boldsymbol{\epsilon}}_{Schering} \\ \hat{\boldsymbol{\epsilon}}'_{BMW}\hat{\boldsymbol{\epsilon}}_{Schering} & \hat{\boldsymbol{\epsilon}}'_{Schering}\hat{\boldsymbol{\epsilon}}_{Schering} \end{pmatrix} \\ &= \begin{pmatrix} 21.524 & 2.107 \\ 2.107 & 12.781 \end{pmatrix}\end{aligned}$$

In addition to the market model (1), the restricted model

$$\begin{pmatrix} r_{BMW,t} \\ r_{Schering,t} \end{pmatrix} = \begin{pmatrix} \beta_{BMW} \\ \beta_{Schering} \end{pmatrix} r_{DAX,t} + \begin{pmatrix} \epsilon_{BMW,t} \\ \epsilon_{Schering,t} \end{pmatrix}$$

was estimated, and resulted in an estimated residual covariance matrix of

$$\hat{\Sigma}_0 = \begin{pmatrix} 21.564 & 2.173 \\ 2.173 & 12.894 \end{pmatrix}.$$

Use the reported results and the likelihood ratio test principle to test for the CAPM in standard form for the two stocks under study ($\alpha = 0.05$).

Analogous to $a)(iii)$ we employ a LR-Test for a SUR-system:

$$\begin{aligned} LR &= T[\ln(\det(\hat{\Sigma}_0)) - \ln(\det(\hat{\Sigma}_1))] \\ &= 244[\ln(273.24) - \ln(270.659)] \\ &= 2.391 \\ \tau_C &= \chi^2_{0.95}(\underbrace{2}_{\alpha_1=\alpha_2=0}) = 5.99 \\ \Rightarrow LR &< 5.99, \end{aligned}$$

i.e., we can not reject the CAPM-hypotheses ($\alpha_1 = \alpha_2 = 0$) for BMW and Schering.