Solution to Problem Set 2, assignment c, i)

To construct the combined efficient set, we have to find the vector of weights \( x_M \) for the \( N \) risky assets, that corresponds to point \((\sigma_M, \mu_M)\).

The efficient line, which is also called capital market line, can be computed by

\[
\max_x \frac{\mu_M - r_f}{\sigma_M} \quad \text{s.t.} \quad 1_N'x_M = 1
\]

which maximizes the slope where \( \mu_M = \mu'x_M \) and \( \sigma_M^2 = \sigma'x_M = x_M'\Sigma x_M \). To solve this, we substitute the add-up constraint into the objective function to obtain an unconstraint maximization problem. To this end define the excess-return vector

\[
\tilde{\mu} = \mu - 1_Nr_f = \begin{pmatrix} \mu_1 - r_f \\ \vdots \\ \mu_N - r_f \end{pmatrix},
\]

and note that

\[ r_f = r_f1_N'x_M, \]

so that we can write

\[
\frac{\mu_M - r_f}{\sigma_M} = \frac{\mu'x_M - r_f1_N'x_M}{\sigma_M} = \frac{\tilde{\mu}'x_M}{(x_M'\Sigma x_M)^{1/2}}
\]

and obtain the unconstraint maximization problem

\[
\max_{x_M} \beta := \frac{\tilde{\mu}'x_M}{(x_M'\Sigma x_M)^{1/2}}.
\]

Thus,

\[
\frac{\partial \beta}{\partial x_M} \overset{\text{product rule}}{=} \tilde{\mu}(x_M'\Sigma x_M)^{-1/2} + \tilde{\mu}'x_M(-1/2)(x_M'\Sigma x_M)^{-3/2} \cdot \frac{2\Sigma x_M}{x_M'\Sigma x_M} = 0 \quad 1_N \times N
\]

\[
\LHS \equiv \tilde{\mu}'x_M (x_M'\Sigma x_M)^{1/2} \quad \overset{\text{LHS}}{=} \begin{pmatrix} \mu \\ \vdots \\ \mu_N \end{pmatrix} \quad \overset{\text{RHS}}{=} \begin{pmatrix} \Sigma^{-1} \tilde{\mu} \\ \vdots \\ \Sigma^{-1} \tilde{\mu} \end{pmatrix} \quad (1)
\]
To solve for $x_M$ we premultiply both sides with $1'_N$. The sum of the LHS is

$$LHS : 1'_N \frac{\tilde{\mu}'x_M}{x_M}'N = \frac{\tilde{\mu}'x_M}{x_M}'N 1'_N x_M = \frac{\tilde{\mu}'x_M}{x_M}'N$$

and the RHS becomes

$$1'_N \Sigma^{-1} \tilde{\mu} \text{ (2),}$$

so that

$$1'_N \Sigma^{-1} \tilde{\mu} = \frac{\tilde{\mu}'x_M}{x_M}'N \Sigma$$

Dividing (1) by (2) we can solve for $x_M$:

$$\frac{\tilde{\mu}'x_M}{x_M}'N \Sigma^{-1} \tilde{\mu} = \Sigma^{-1} \tilde{\mu}$$

$$x_M = \frac{\tilde{\mu}^{-1} \tilde{\mu}}{1'_N \Sigma^{-1} \tilde{\mu}} = \frac{\Sigma^{-1} \tilde{\mu}}{1'_N \Sigma^{-1} \tilde{\mu}} = \frac{\Sigma^{-1} \tilde{\mu}}{1'_N \Sigma^{-1} \tilde{\mu}} = \frac{1}{b - cr_f} \Sigma^{-1} \tilde{\mu}$$

$$= \mu_M = \tilde{\mu}'x_M \text{ for details, see script} \ a - br_f$$

$$\Rightarrow \sigma^2_M = x_M' \Sigma x_M = \frac{1}{(b - cr_f)^2} (\mu' - 1'_N r_f) \Sigma^{-1} (\mu - 1'_N r_f)$$

$$= \frac{1}{(b - cr_f)^2} (\mu' \Sigma^{-1} \mu - \mu' \Sigma^{-1} 1'_N r_f - r_f \Sigma^{-1} \mu + r_f^2 \Sigma^{-1} 1'_N) = \frac{a - 2br_f + cr_f^2}{(b - cr_f)^2}$$

Answer to the problem:

$$x_M = \frac{1}{11/9 - 5/9} \begin{pmatrix} 5/9 & -1/9 \\ -1/9 & 2/9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

That is, 50% are invested in the first risky asset and the other 50% are invested in the second asset regarding the tangency portfolio.

$$\mu_M = \frac{26/9 - 11/9}{11/9 - 5/9} = 2.5$$

$$\sigma^2_M = \frac{26/9 - 2 * 11/9 + 5/9}{(1/9 - 5/9)^2} = 2.25$$

Note: Regarding the exam, you also have to state the general formulas!