Solution to Problem Set 2, assignment c, i)

To construct the combined efficient set, we have to find the vector of weights x_M for the N risky assets, that corresponds to point (σ_M, μ_M).

The efficient line, which is also called capital market line, can be computed by

$$\max_x \frac{\mu_M - r_f}{\sigma_M} \quad \text{s.t.} \quad 1'_N x_M = 1$$

which maximizes the slope where $\mu_M = \mu' x_M$ and $\sigma_M^2 = x'_M \Sigma x_M$. To solve this, we substitute the add-up constraint into the objective function to obtain an unconstraint maximization problem. To this end define the excess-return vector

$$\underbrace{\tilde{\mu}}_{N\times 1} = \mu - 1_N r_f = \begin{pmatrix} \mu_1 - r_f \\ \vdots \\ \mu_N - r_f \end{pmatrix},$$

and note that

$$r_f = r_f 1'_N x_M,$$

so that we can write

$$\frac{\mu_M - r_f}{\sigma_M} = \frac{\mu' x_M - r_f 1'_N x_M}{\sigma_M} = \frac{\tilde{\mu}' x_M}{(x'_M \Sigma x_M)^{1/2}}$$

and obtain the unconstraint maximization problem

$$\max_{x_M} \beta := \frac{\tilde{\mu}' x_M}{(x'_M \Sigma x_M)^{1/2}}.$$

Thus,

$$\frac{\partial \beta}{\partial x_M} \overset{\text{product rule}}{=} \tilde{\mu}(x'_M \Sigma x_M)^{-1/2} + \tilde{\mu}' x_M (-1/2)(x'_M \Sigma x_M)^{-3/2} \underbrace{2\Sigma x_M}_{\frac{\partial x'_M \Sigma x_M}{\partial x_M}} = \underbrace{0}_{N\times 1}$$

$$\Leftrightarrow \frac{\tilde{\mu}' x_M}{(x'_M \Sigma x_M)^{3/2}} \Sigma x_M = \frac{\tilde{\mu}}{(x'_M \Sigma x_M)^{1/2}}$$

$$\Leftrightarrow \underbrace{\underbrace{\frac{\tilde{\mu}' x_M}{x'_M \Sigma x_M}}_{1\times 1} \underbrace{x_M}_{N\times 1}}_{\text{LHS}} = \underbrace{\underbrace{\Sigma^{-1}}_{N\times N} \underbrace{\tilde{\mu}}_{N\times 1}}_{\text{RHS}} \quad (1)$$

To solve for x_M we premultiply both sides with $1_N'$. The sum of the LHS is

$$LHS: 1_N' \frac{\tilde{\mu}' x_M}{x_M' \Sigma x_M} x_M = \frac{\tilde{\mu}' x_M}{x_M' \Sigma x_M} \underbrace{1_N' x_M}_{=1} = \frac{\tilde{\mu}' x_M}{x_M' \Sigma x_M}$$

and the RHS becomes

$$1_N' \Sigma^{-1} \tilde{\mu} \quad (2),$$

so that

$$1_N' \Sigma^{-1} \tilde{\mu} = \frac{\tilde{\mu}' x_M}{x_M' \Sigma x_M}$$

Dividing (1) by (2) we can solve for x_M:

$$\begin{aligned}
\frac{\frac{\tilde{\mu}' x_M}{x_M' \Sigma x_M}}{1_N' \Sigma^{-1} \tilde{\mu}} x_M &= \frac{\Sigma^{-1} \tilde{\mu}}{1_N' \Sigma^{-1} \tilde{\mu}} \\
x_M &= \frac{\Sigma^{-1} \tilde{\mu}}{1_N' \Sigma^{-1} \tilde{\mu}} = \frac{\Sigma^{-1} \tilde{\mu}}{1_N' \Sigma^{-1} (\mu - 1_N r_f)} = \frac{\Sigma^{-1} \tilde{\mu}}{\underbrace{1_N' \Sigma^{-1} \mu}_{=b} - \underbrace{1_N' \Sigma^{-1} 1_N}_{=c} r_f} = \frac{1}{b - c r_f} \Sigma^{-1} \tilde{\mu} \\
\Rightarrow \mu_M &= \mu' x_M \overset{\text{for details, see script}}{=} \frac{a - b r_f}{b - c r_f} \\
\Rightarrow \sigma_M^2 &= x_M' \Sigma x_M = \frac{1}{(b - c r_f)^2} (\mu' - 1_N' r_f) \Sigma^{-1} (\mu - 1_N r_f) \\
&= \frac{1}{(b - c r_f)^2} (\underbrace{\mu' \Sigma^{-1} \mu}_{=a} - \underbrace{\mu' \Sigma^{-1} 1_N}_{=b} r_f - r_f \underbrace{1_N' \Sigma^{-1} \mu}_{=b} + r_f^2 \underbrace{1_N \Sigma^{-1} 1_N}_{c}) = \frac{a - 2b r_f + c r_f^2}{(b - c r_f)^2}.
\end{aligned}$$

Answer to the problem:

$$x_M = \frac{1}{11/9 - 5/9} \begin{pmatrix} 5/9 & -1/9 \\ -1/9 & 2/9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

That is, 50% are invested in the first risky asset and the other 50% are invested in the second asset regarding the tangency portfolio.

$$\begin{aligned}
\mu_M &= \frac{26/9 - 11/9}{11/9 - 5/9} = 2.5 \\
\sigma_M^2 &= \frac{26/9 - 2 * 11/9 + 5/9}{(1/9 - 5/9)^2} = 2.25
\end{aligned}$$

Note: Regarding the exam, you also have to state the general formulas!