Problem Set 1: Expected Utility

a) A person with initial wealth $w_0 > 0$ has a utility function of the form $U(W) = \ln(W)$. She is offered the opportunity to bet on the flip of a coin that has a probability of p of coming up heads. If she bets x ($x \geq 0$), she will have $w_0 + x$ if heads comes up and $w_0 - x$ if tails comes up.

   i) Solve for the optimal $x^\star$ as a function of p.

   ii) Does the investor in this example exhibit decreasing absolute risk aversion?

b) A person with initial wealth $w_0 = 4$ € has a utility function of the form $U(W) = \sqrt{W}$ and owns a lottery ticket that will be worth $x = 0$ € with probability $p = 0.5$ and worth $x = 12$ € otherwise.

   What is the lowest price π_{min} at which she would sell the ticket in order to avoid gambling?

c) Consider two agents A and B with quadratic utility functions

$$U_i(W) = W - \frac{\beta_i}{2}W^2, \quad i = A, B \tag{1}$$

   where $\beta_A = 0.5$ and $\beta_B = 0.75$. Both agents have initial wealth $w_0 = 1$. They have to invest this wealth in a portfolio of two risky assets with *net returns*[1] R_1 and R_2.
   It is known that $E[R_1] = \mu_1 = 0.25, E[R_2] = \mu_2 = 0.05, Var[R_1] = \sigma_1^2 = 1$ and $Var[R_2] = \sigma_2^2 = 1.5$. The returns are uncorrelated, i.e., $\rho_{12} = 0$.

   i) Show that agent B is more risk averse than agent A according to the Arrow-Pratt measure of absolute risk aversion.

   ii) Derive the optimal portfolio weights of the first asset for both individuals. Let these be denoted by $x_A^\star$ and $x_B^\star$ for agents A and B, respectively.

[1] If P_t is the price of the asset at time t, then the *net returns* is $r_t = (P_t - P_{t-1})/P_{t-1}$, and the *gross returns* is $R_t = 1 + r_t = P_t/P_{t-1}$

iii) The results show that the more risk averse individual invests less in the asset with the lower variance. Is this in conflict with intuition?

d) A person with initial wealth $w_0 > 0$ and utility function $U(W) = \ln(W)$ has two investment alternatives: A risk-free asset, which pays no interest (e.g. money), and a risky asset yielding a net return equal to $r_1 < 0$ with probability p and equal to $r_2 > 0$ with probability $1 - p$ in the next period. Denote the fraction of initial wealth to be invested in the risky asset by x. Find the fraction x which maximizes the expected utility of wealth in the next period. Denote this solution by $x^\star$. What is the condition for $x^\star > 0$?

e) Now consider a problem similar to that in (d), but the utility function of the investor is $U(W) = -\exp(-cW), c > 0$, and the return of the risky asset is normally distributed with mean μ and variance σ^2.

Find the fraction x which maximizes the expected utility of wealth in the next period.

Hint: Remember the moment-generating function of the normal distribution:

$$M_X(t) = E[e^{tX}] = \exp[t(\mu + t\sigma^2/2)].$$

f) An investor with $U(W) = \ln(W)$ has to decide between two lotteries:

$$\text{payout } L_1\text{:} \begin{cases} 1 \text{ €} & \text{with probability } 0.8 \\ 100 \text{ €} & \text{else} \end{cases}$$

$$\text{payout } L_2\text{:} \begin{cases} 10 \text{ €} & \text{with probability } 0.99 \\ 1000 \text{ €} & \text{else} \end{cases},$$

with $E[L_1] = 20.8, Var[L_1] = 1568.16, E[L_2] = 19.9, Var[L_2] = 9702.00$.

Show that the choices made upon expected utility theory or mean-variance analysis do not coincide in this case!

g) State some conditions when mean-variance analysis is consistent with expected utility theory.