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Risk Estimation and Spurious Seasonality

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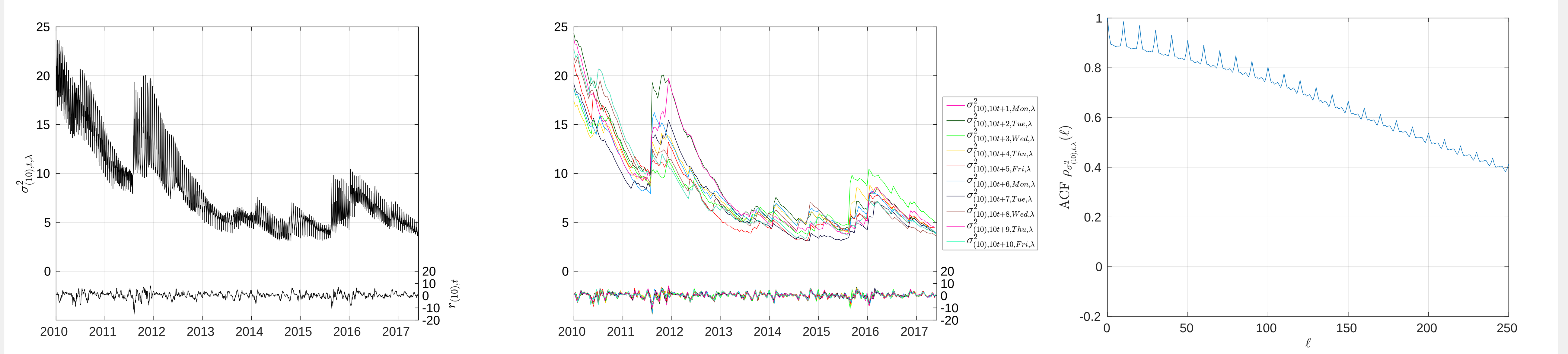
Abstract

An important task in financial and insurance risk management is the specification of the regulatory risk capital required to cover potential future losses. To quantify that buffer, institutions are obliged to estimate and predict certain risk measures. To do so, regulation typically prescribes both forecasting horizon and reporting frequency, but does not specify the sampling scheme for the data used for estimation. In this paper, we demonstrate that the scheme with which data are sampled crucially affects variance and, thus, risk estimates. We show that temporal sequences of variance estimates, in cases where the reporting frequency is higher than the sampling frequency of non-overlapping returns, will suffer from spurious seasonality. The implications of spurious seasonality for risk management are discussed in the context of the Basel III rules for banking regulation. We present a boundary-corrected, exponentially-weighted moving-average version of a two-scale variance estimator to overcome the problem of spurious seasonality.

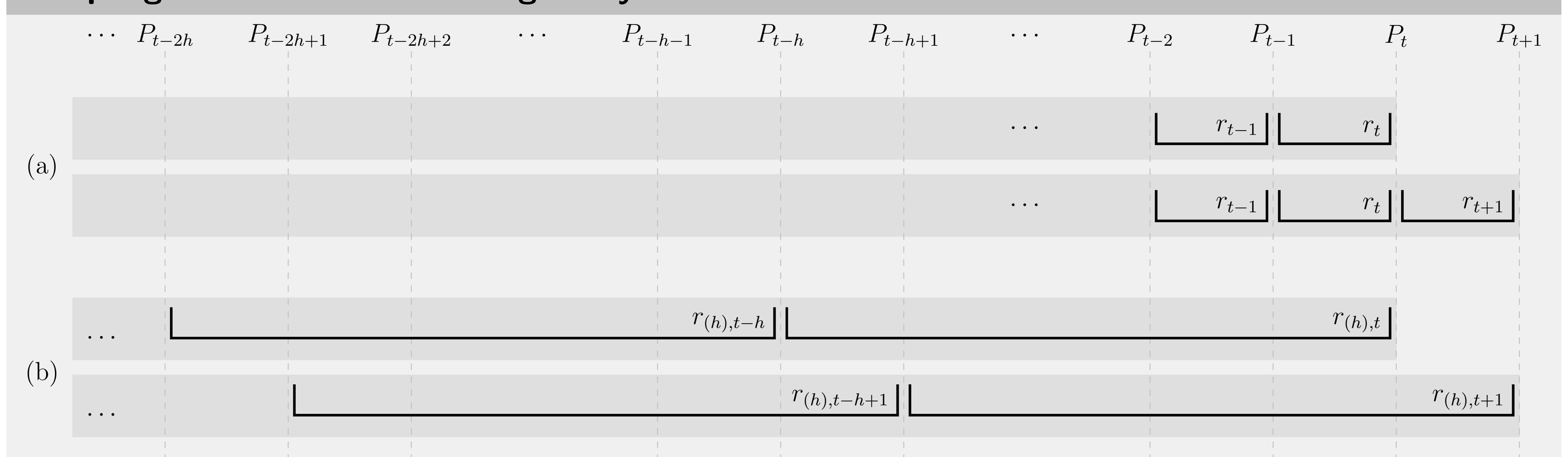
EWMA variances of the DJIA based on non-overlapping 10-day log-returns (01-Jan-2010 to 02-Jun-2017; Window length: 100 bi-weekly returns)

- Daily sequence of estimated variances $(\sigma^2_{(10),t,\lambda})_{t\in\mathbb{Z}}$, bi-weekly sequences of estimated variances $(\sigma^2_{(10),10t+\tau,\lambda})_{t\in\mathbb{Z}}$, for $1 \le \tau \le 10$ and the ACF of the daily sequence of estimated variances.



- We analyze **time series properties of sequences of risk estimates** with a special focus on variances.
- We determine consequences of the **implicit overlap** if one reports and **estimates** risk measures on a **higher frequency** than the **aggregation horizon** of the non-overlapping return data used for each estimation. ⇒ **Spurious seasonality in temporal sequences of estimated variances**.

Sampling schemes for estimating h-day risk measures



- EWMA variance with **daily** returns (a):

$$\sigma^2_{(1),t,\lambda} = h\frac{1-\lambda^{\frac{1}{h}}}{1-\lambda^{\Delta}}\sum_{\tau=0}^{h\Delta-1}\lambda^{\frac{\tau}{h}}(r_{t-\tau}-\mu_{(1),t,\lambda})^2$$

- EWMA variance with **non-overlapping h-day** returns (b):

$$\boxed{\sigma^2_{(h),t,\lambda} = \frac{1-\lambda}{1-\lambda^{\Delta}}\sum_{\delta=0}^{\Delta-1}\lambda^{\delta}(r_{(h),t-h\delta}-\mu_{(h),t,\lambda})^2}$$

Variance estimators as quadratic forms

- δ-periods **vector of daily returns** up to time t: $\mathbf{r}_{t,\delta} := [r_{t-\delta+1}, r_{t-\delta+2}, \dots, r_{t-1}, r_t]'$
- **h-day returns**: $r_{(h),t} = \ln(P_t) - \ln(P_{t-h}) = \sum_{j=0}^{h-1} r_{t-j} = \mathbf{1}_h'\mathbf{r}_{t,h}$
- Δ-periods **vector of non-overlapping h-day returns** up to time t (with $\mathbf{H} = \mathbf{I}_\Delta \otimes \mathbf{1}_h$):

$$\mathbf{r}_{(h),t,\Delta} = \left[r_{(h),t-h(\Delta-1)}, r_{(h),t-h(\Delta-2)}, \dots, r_{(h),t-h}, r_{(h),t}\right]' = \mathbf{H}'\mathbf{r}_{t,h\Delta}$$

- Variance estimators as **quadratic forms** in $\mathbf{r}_{t,h\Delta}$ (e.g., sample or EWMA variance estimator):

$$\boxed{\sigma^2_{(h),t} = \mathbf{r}'_{t,h\Delta}\mathbf{Q}\mathbf{r}_{t,h\Delta}}$$

- Define the two matrices $\mathbf{K}, \mathbf{L} \in \mathbb{R}^{h\Delta+\ell\times h\Delta}$ for $\ell \ge 0$:

$$\mathbf{K} = \left[\mathbf{0}_{(h\Delta\times\ell)}, \mathbf{I}_{h\Delta}\right]', \qquad \mathbf{L} = \left[\mathbf{I}_{h\Delta}, \mathbf{0}_{(h\Delta\times\ell)}\right]'$$

- For **quadratic form based variance estimators** it follows:

$$\boxed{\sigma^2_{(h),t} = \mathbf{r}'_{t,h\Delta}\mathbf{Q}\mathbf{r}_{t,h\Delta} = \mathbf{r}'_{t,h\Delta+\ell}\mathbf{K}\mathbf{Q}\mathbf{K}'\mathbf{r}_{t,h\Delta+\ell}}$$

- For the **lagged (by ℓ days) variance estimator** it follows:

$$\boxed{\sigma^2_{(h),t-\ell} = \mathbf{r}'_{t-\ell,h\Delta}\mathbf{Q}\mathbf{r}_{t-\ell,h\Delta} = \mathbf{r}'_{t,h\Delta+\ell}\mathbf{L}\mathbf{Q}\mathbf{L}'\mathbf{r}_{t,h\Delta+\ell}}$$

Theorem

Let $(x_t)_{t\in\mathbb{Z}}$ be a stochastic process with $\mathbb{E}(|x_t|^i) < \infty$, for $t \in \mathbb{Z}$ and $i \le 4$. For $t_1, t_2, t_3, t_4 \in \mathbb{Z}$ with $\forall i,j \in \{t_1,t_2,t_3,t_4\}$: $i \ne j$, we assume

$$\mathbb{E}(x_{t_1}) = 0, \qquad \mathbb{E}(x_{t_1}x_{t_2}x_{t_3}x_{t_4}) = 0, \qquad \mathbb{E}(x_{t_1}^2x_{t_2}x_{t_3}) = 0, \qquad \text{and} \qquad \mathbb{E}(x_{t_1}^3x_{t_2}) = 0. \qquad (*)$$

Let $\mathbf{X} = [x_1, \dots, x_n]'$ and define $\mathbf{X}^{2\odot} = \mathbf{X}\odot\mathbf{X} = [x_1^2,\dots,x_n^2]'$. Furthermore, define vector $\boldsymbol{\mu}_{\mathbf{X}^{2\odot}} \in \mathbb{R}^{n\times 1}$ and matrices $\boldsymbol{\Sigma}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{X}^{2\odot}} \in \mathbb{R}^{n\times n}$ by

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \mathbb{E}(\mathbf{X}\mathbf{X}'), \quad \boldsymbol{\mu}_{\mathbf{X}^{2\odot}} = \mathbb{E}(\mathbf{X}^{2\odot}) \quad \text{and} \quad \boldsymbol{\Sigma}_{\mathbf{X}^{2\odot}} = \mathbb{E}(\mathbf{X}^{2\odot}\mathbf{X}^{2\odot\prime}) - \boldsymbol{\mu}_{\mathbf{X}^{2\odot}}\boldsymbol{\mu}'_{\mathbf{X}^{2\odot}},$$

respectively. Then, for symmetric matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, we have

$$\boxed{\mathrm{Cov}(\mathbf{X}'\mathbf{A}\mathbf{X}, \mathbf{X}'\mathbf{B}\mathbf{X}) = \mathrm{tr}(\mathbf{C}(\boldsymbol{\Sigma}_{\mathbf{X}^{2\odot}} + \boldsymbol{\mu}_{\mathbf{X}^{2\odot}}\boldsymbol{\mu}'_{\mathbf{X}^{2\odot}})) - \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{X}})\mathrm{tr}(\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{X}}),}$$

with $\mathbf{C} = \mathbf{a}\mathbf{b}' + 2\mathbf{A}\odot\mathbf{B}\odot(\mathbf{1}_n\mathbf{1}_n' - \mathbf{I}_n)$, where $\mathbf{a} = \mathrm{diag}(\mathbf{A}) = (\mathbf{A}\odot\mathbf{I}_n)\mathbf{1}_n$ and $\mathbf{b} = \mathrm{diag}(\mathbf{B}) = (\mathbf{B}\odot\mathbf{I}_n)\mathbf{1}_n$.

Corollary

Let $(r_t)_{t\in\mathbb{Z}}$ be a weak white noise process fulfilling the moment conditions $(*)$ stated in the Theorem. Moreover, let $\sigma^2 = \mathrm{Var}(r_t) = \mathbb{E}(r_t^2)$ and $\mathbf{r}^{2\odot}_{t,h\Delta+\ell} = \mathbf{r}_{t,h\Delta+\ell}\odot\mathbf{r}_{t,h\Delta+\ell}$, and define vector $\boldsymbol{\mu}_{\mathbf{r}^{2\odot}_{t,h\Delta+\ell}} \in \mathbb{R}^{h\Delta+\ell\times 1}$ and matrix $\boldsymbol{\Sigma}_{\mathbf{r}^{2\odot}_{t,h\Delta+\ell}} \in \mathbb{R}^{h\Delta+\ell\times h\Delta+\ell}$ by

$$\boldsymbol{\mu}_{\mathbf{r}^{2\odot}_{t,h\Delta+\ell}} = \mathbb{E}(\mathbf{r}^{2\odot}_{t,h\Delta+\ell}) \qquad \text{and} \qquad \boldsymbol{\Sigma}_{\mathbf{r}^{2\odot}_{t,h\Delta+\ell}} = \mathbb{E}(\mathbf{r}^{2\odot}_{t,h\Delta+\ell}\mathbf{r}^{2\odot\prime}_{t,h\Delta+\ell}) - \boldsymbol{\mu}_{\mathbf{r}^{2\odot}_{t,h\Delta+\ell}}\boldsymbol{\mu}'_{\mathbf{r}^{2\odot}_{t,h\Delta+\ell}},$$

respectively. Then, considering variance estimates of the form $\sigma^2_{(h),t} = \mathbf{r}'_{t,h\Delta}\mathbf{Q}\mathbf{r}_{t,h\Delta}$, the autocovariance of the series $(\sigma^2_{(h),t})_{t\in\mathbb{Z}}$ for $\ell \ge 0$ is given by

$$\boxed{\gamma_{\sigma^2_{(h),t}}(\ell) = \mathrm{tr}\left(\mathbf{C}\boldsymbol{\Sigma}_{\mathbf{r}^{2\odot}_{t,h\Delta+\ell}}\right) + 2\sigma^4\left(\mathrm{tr}(\mathbf{K}\mathbf{Q}\mathbf{K}'\mathbf{L}\mathbf{Q}\mathbf{L}') - \mathbf{a}'\mathbf{b}\right),}$$

with $\mathbf{C} = \mathbf{a}\mathbf{b}' + 2(\mathbf{K}\mathbf{Q}\mathbf{K}')\odot(\mathbf{L}\mathbf{Q}\mathbf{L}')\odot(\mathbf{1}_{h\Delta+\ell}\mathbf{1}'_{h\Delta+\ell} - \mathbf{I}_{h\Delta+\ell})$, where $\mathbf{a} = \mathrm{diag}(\mathbf{K}\mathbf{Q}\mathbf{K}')$ and $\mathbf{b} = \mathrm{diag}(\mathbf{L}\mathbf{Q}\mathbf{L}')$.

Remark

The following processes satisfy the moment conditions $(*)$ of the Theorem:

- **Gaussian white noise process**, if $\mu = 0$
- **GARCH(p,q) processes** $(x_t)_{t\in\mathbb{Z}}$ with $x_t = \sigma_t\epsilon_t$ and $\sigma_t^2 = \alpha_0 + \sum_{i=1}^q \alpha_i x_{t-i}^2 + \sum_{i=1}^p \beta_i\sigma^2_{t-i}$, if
  - Innovations $(\epsilon_t)_{t\in\mathbb{Z}}$: Symmetrically distributed i.i.d. random variables such that odd moments are zero
  - Existence and finiteness of first four moments of $(x_t)_{t\in\mathbb{Z}}$ (Bollerslev 1986; He and Teräsvirta 1999; Ling and McAleer 2002a)

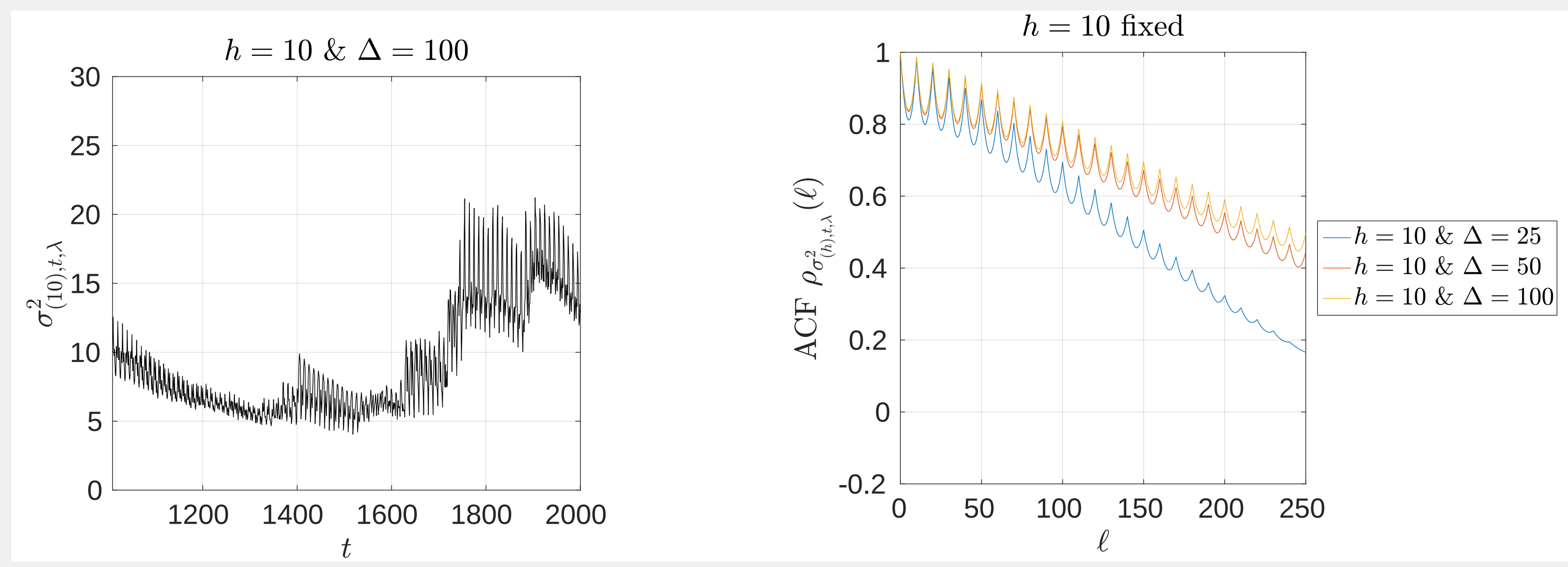
Setup for the visualizations and the simulation study

- Assume a GARCH(1,1) process for the **daily return process** $(r_t)_{t\in\mathbb{Z}}$ with normal innovations:

$$r_t = \sigma_t\epsilon_t, \qquad \text{and} \qquad \sigma_t^2 = 0.01 + 0.05r_{t-1}^2 + 0.94\sigma_{t-1}^2$$

- **Aggregation frequency**: $h = 10$ (Basel III rules BCBS (2016))
- **Rolling-window size**: $\Delta = 100$ ($\Rightarrow h\Delta = 1000$ daily observations; approximately four years of data)
- **Overall sample size**: $8 \times 250 = 2000$ daily return observations; approximately 8 years of data

Sequence of EWMA variance estimates $(\sigma^2_{(10),t,\lambda})_{t\in\mathbb{Z}}$ and its ACF



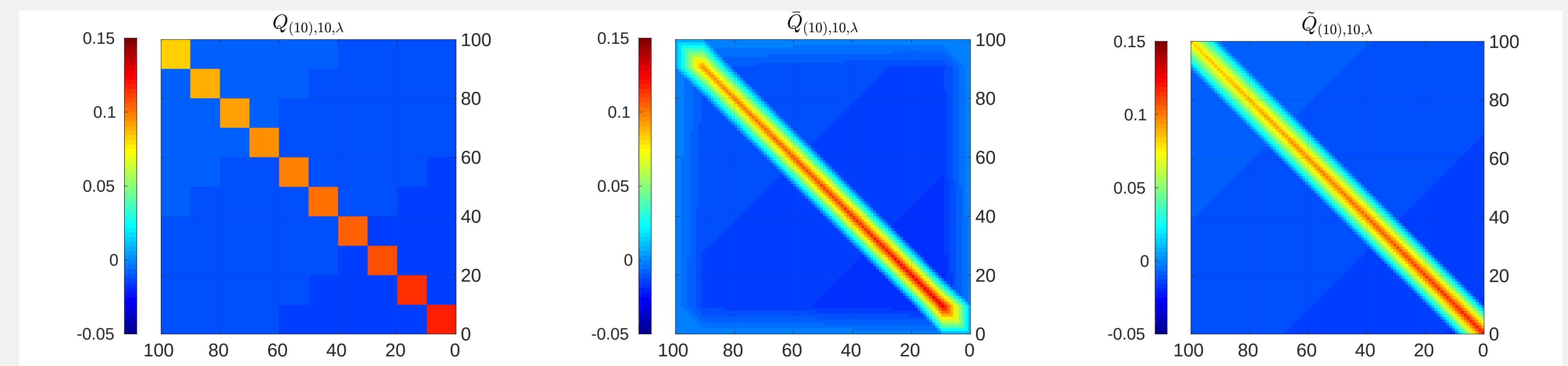
Variance estimators based on overlapping returns

- **Two-scales sample variance**, proposed in the (ultra-)high-frequency context (Zhang et al. 2005):

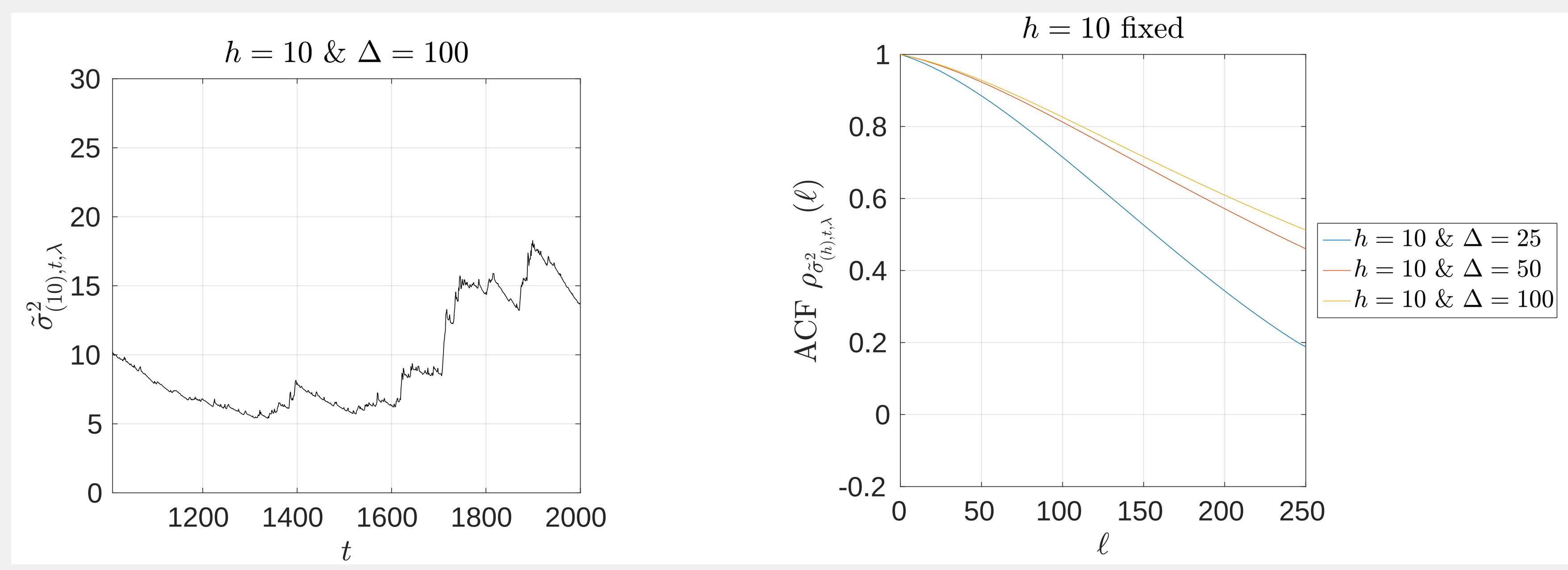
$$\boxed{\tilde{\sigma}^2_{(h),t} = \frac{1}{h}\mathbf{r}'_{t,h\Delta}\mathbf{Q}_{(h),\Delta}\mathbf{r}_{t,h\Delta} + \frac{1}{h}\sum_{j=1}^{h-1}\mathbf{r}'_{t-j,h(\Delta-1)}\mathbf{Q}_{(h),\Delta-1}\mathbf{r}_{t-j,h(\Delta-1)}}$$

- We introduce a **boundary-corrected EWMA version of the two-scales variance estimator**.

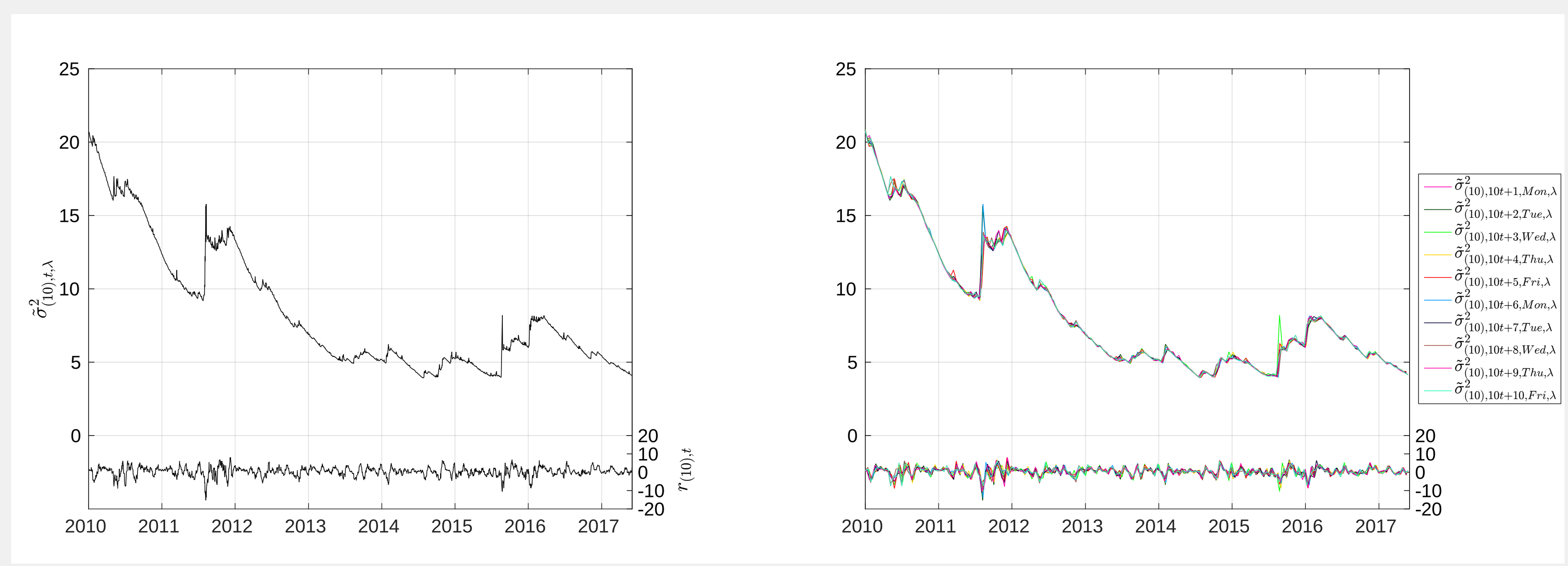
Visualization of quadratic form-based variance estimators



Sequence of boundary-corrected two-scales EWMA variances $(\tilde{\sigma}^2_{(10),t,\lambda})_{t\in\mathbb{Z}}$ and its ACF



Boundary-corrected two-scales EWMA variances of the DJIA



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