Slides for Risk Management

Introduction to copulas

Groll

Seminar für Finanzökonometrie

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1. Copula theory
    - Pitfalls of correlation
    - Introduction to copula theory
    - Simulation
    - *VaR* in terms of copulas

Linear correlation

Definition

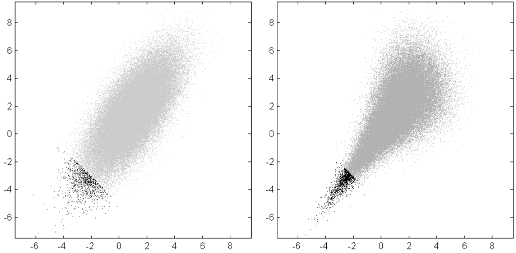
The **linear correlation coefficient** between two random variables X and Y is

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\sigma_X^2 \sigma_Y^2}},$$

where $Cov(X,Y)$ denotes the covariance between X and Y, and σ_X^2, σ_Y^2 denote the variances of X and Y.

Same correlation, different dependency

- there is more to dependence than what can be captured by linear correlation
- example, showing simulated values with **equal margins** and **equal estimated correlation coefficient** $\hat{\rho} = 0.701$, but different overall dependence structure:



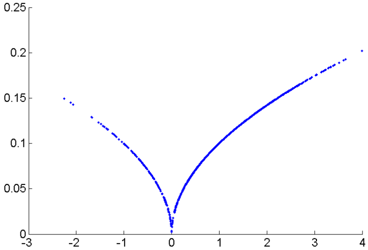
Pitfalls of linear correlation

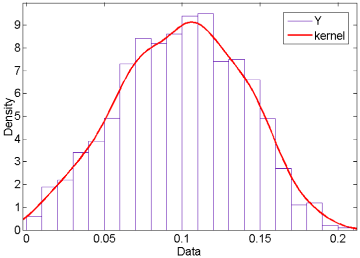
given marginal distributions $X \sim F_X$ and $Y \sim F_Y$,...

- ... the linear correlation coefficient does **not** completely **determine the joint distribution**
- ... it is in general **not possible** to construct a joint distribution of the margins with **arbitrary associated correlation coefficient**
    - given that F_X can be obtained by application of a non-linear deterministic function f on Y, $f(Y) \sim F_X$, the linear correlation coefficient ρ_{max} associated with $(f(Y), Y)$ is in general lower than 1, $\rho_{max} < 1$, since linear correlation can only capture **linear dependence**
    - nevertheless, values of ρ that are higher than ρ_{max} can not be obtained for marginal distributions F_X and F_Y
- ... the **worst** case VaR_α of a portfolio of X and Y does **not** necessarily occur when the **correlation is maximal**

Pitfalls of linear correlation

$$X \sim \mathcal{N}(1,1), \quad Y = \sqrt{\left| exp\left(\frac{X}{100}\right) - 1 \right|}, \quad \hat{\rho} = 0.807$$





- there is no way to combine marginal distributions F_X and F_Y in a joint distribution with correlation higher than $\hat{\rho}$

Probability integral transformation

Theorem

Let X be a univariate random variable with distribution function F_X. Let F_X^{-1} be the quantile function of F_X, i.e.

$$F_X^{-1}(\alpha) = \inf\{x | F_X(x) \geq \alpha\},$$

$\alpha \in (0, 1)$. *Then:*

- *For any standard-uniformly distributed $U \sim \mathbb{U}[0, 1]$ we have $F_X^{-1}(U) \sim F_X$. This gives a simple method for **simulating random variables** with distribution function F.*
- *If F_X is continuous, then the random variable $F_X(X)$ is standard-uniformly distributed, i.e. $F_X(X) \sim \mathbb{U}[0, 1]$. It is called **probability integral transform**.*

Proof

Proof.

Let X be distributed with continuous and invertible cumulative distribution function F_X. Then

$$\mathbb{P}(F_X^{-1}(U) \leq x) = \mathbb{P}(U \leq F_X(x)) = F_X(x),$$

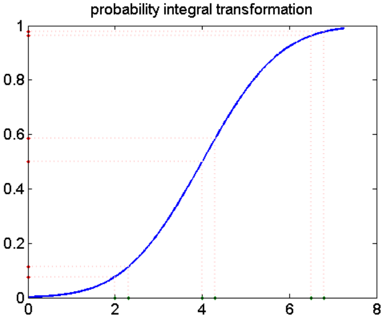
and

$$\mathbb{P}(F_X(X) \leq u) = \mathbb{P}(X \leq F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u.$$

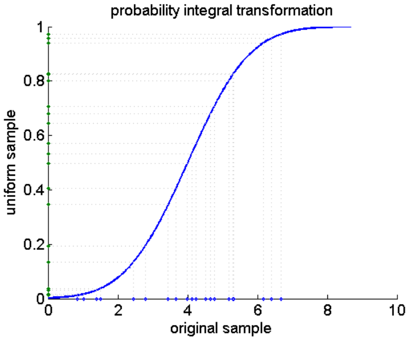
□

Probability integral transformation

- because of the comparatively steeper slope in areas with higher probability, distances between dense points get stretched apart



Probability integral transformation



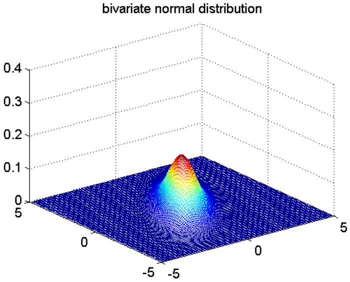
Joint bivariate normal distribution

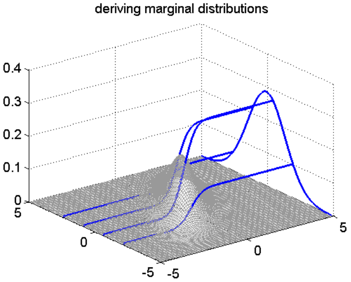
- given that (X, Y) follow a bivariate normal distribution with density function

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right)$$

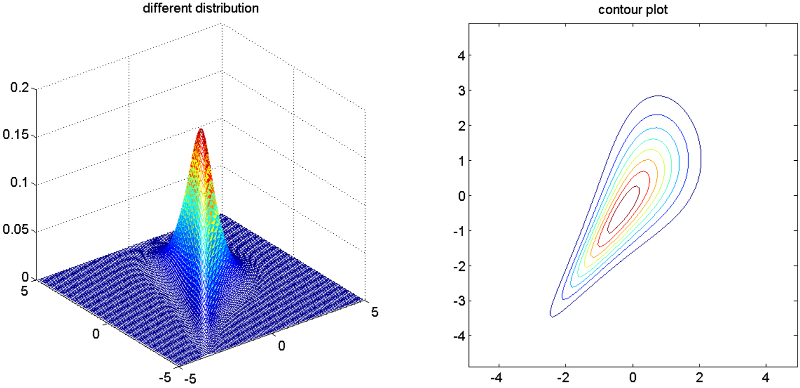
  each variable itself follows a one-dimensional normal distribution: $X \sim \mathcal{N}\left(\mu_X, \sigma_X^2\right)$ and $Y \sim \mathcal{N}\left(\mu_Y, \sigma_Y^2\right)$
- marginal distributions are obtained by integrating out the second variable

Marginal distribution



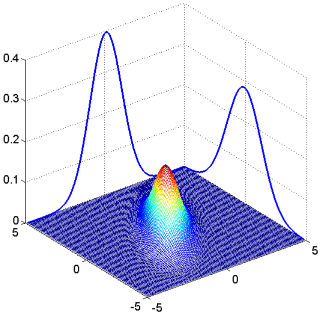


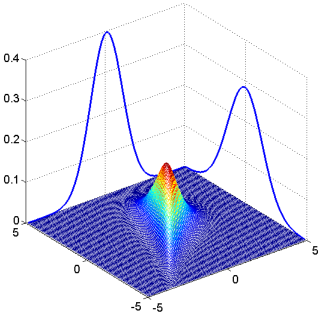
Comparison with other distributions



Comparison with other distributions

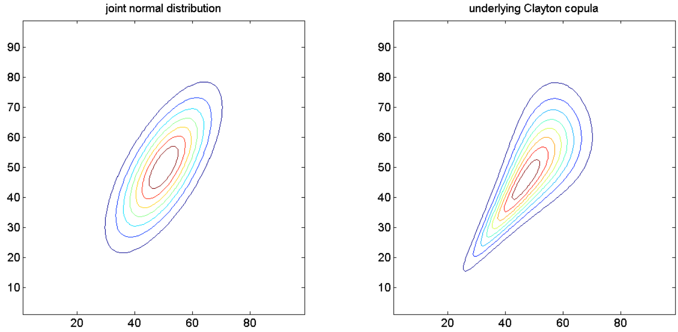
- equal marginal distributions





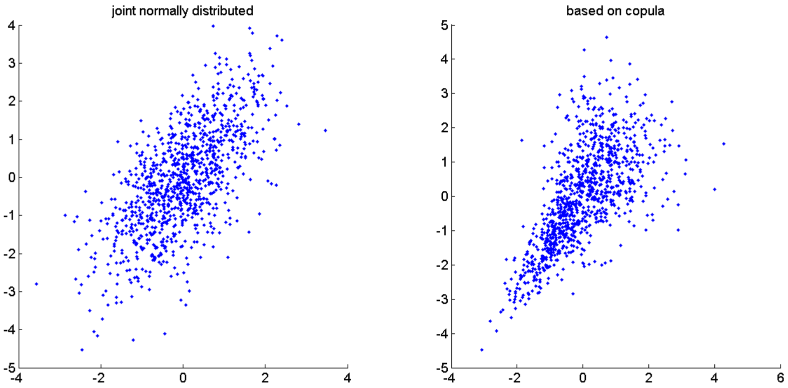
Comparison with other distributions

- contour plots



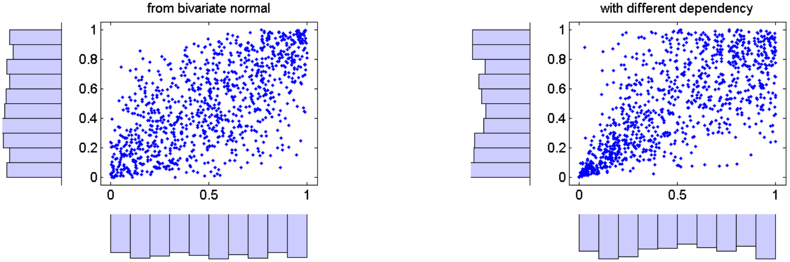
Comparison with other distributions

- same correlation: $\hat{\rho} \approx 0.7$



Translate to unit square

- instead of (X, Y), examine $(F_X(X), F_Y(Y))$:



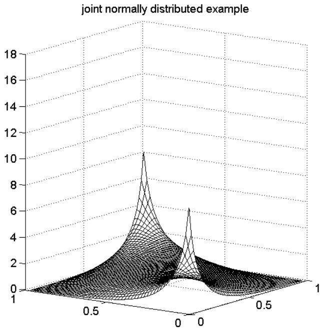
- note: both $F_X(X)$ and $F_Y(Y)$ are uniformly distributed

Intuition

- you can think about a joint probability distribution as assigning probabilities to events measured in absolute terms: "probability of both returns being higher than 3%", "probability of stock A achieving more than 2%, with stock B simultaneously decreasing more than -2%",...
- application of F_X is the opposite direction of the quantile function F_X^{-1}: instead of mapping given quantile probabilities to values in absolute terms, $F_X(x)$ maps absolute value x to its associated quantile probability
- distribution of $(F_X(X), F_Y(Y))$ entails probability information about dependency in "quantile world": "probability of both returns being within the 5% highest realizations", "probability of stock A achieving one of the 37% highest realizations, with stock B simultaneously being within its 15% lowest realizations"

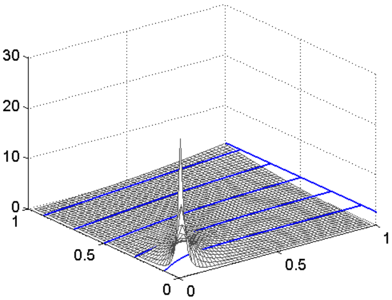
Densities on unit square

- both densities have uniformly distributed margins: information contained in the marginal distributions is excluded, and new marginal distributions can be interpreted as quantiles
- remaining: information about **dependence structure** only



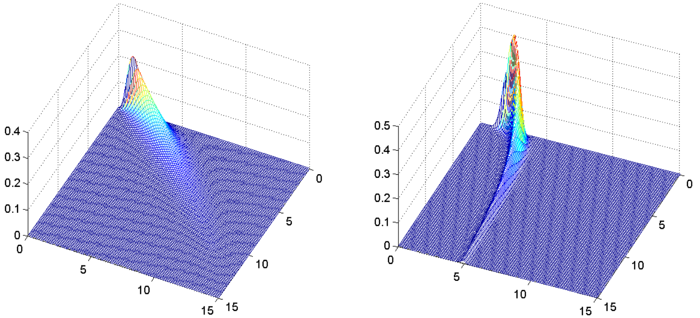
Marginal distributions

- probability integral transformations generate uniform margins

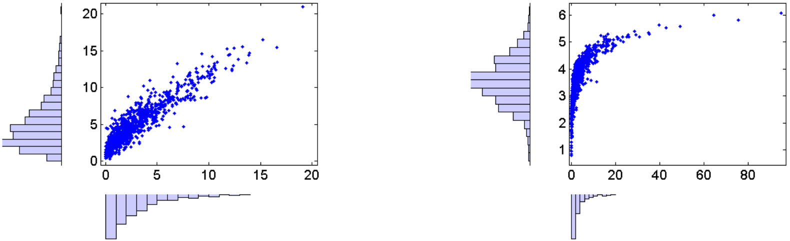


Comparing bivariate distributions

- comparing dependence structures can be difficult

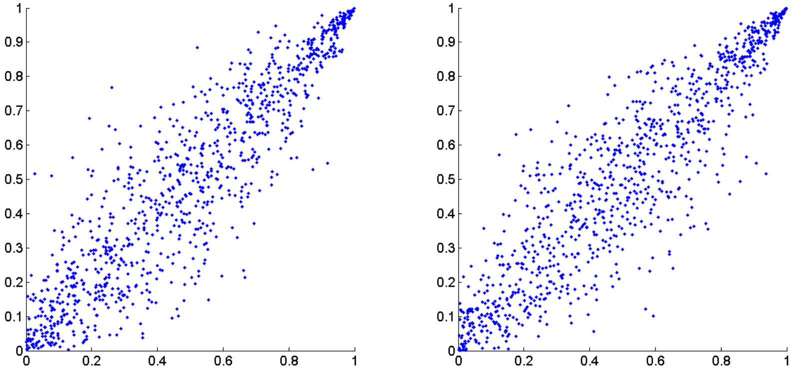


Different margins



Comparison on unit square

- excluding the information contained in the margins shows equal dependence structures



Copula definition

Definition

Let $\mathbf{U} = (U_1, \ldots, U_d)$ be a d-dimensional random vector with standard-uniformly distributed margins

$$U_j \sim \mathbb{U}[0,1] \text{ for } j = 1, ..., d.$$

Then the d-dimensional distribution function $\mathbf{F_U} : [0,1]^d \to [0,1]$ of $\mathbf{U}$ is called a **copula** and will be denoted by $\mathbf{C_U}$. Furthermore, the probability distribution function of $\mathbf{U} = (U_1, \ldots, U_d)$ is called **copula density function** and will be denoted by $\mathbf{c_U}$. It is given by

$$\mathbf{c_U}(u_1, \ldots, u_d) = \frac{\partial^d \mathbf{C}(u_1, \ldots, u_d)}{\partial u_1 \cdots \partial u_d}.$$

Note: in most cases we will restrict ourselves to the two-dimensional case, where visualizations of copulas are still possible. Nevertheless, extensions to higher dimensions are usually possible and straightforward.

Properties of copulas

A d-dimensional copula is a function $\mathbf{C} : I^d \to [0,1]$ with the following properties:

1. $\mathbf{C}$ is grounded: $\mathbf{C}(u_1, \ldots, u_{j-1}, 0, u_j, \ldots, u_d) = 0$, $1 \leq j \leq d$
2. Its one-dimensional margins are the identity function: $\mathbf{C}(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$, $1 \leq j \leq d$
3. It is increasing in each component: $\mathbf{C}(u_1, \ldots, u_j, \ldots, u_d) \leq \mathbf{C}(u_1, \ldots, u_j + \epsilon, \ldots, u_d)$, $1 \leq j \leq d$, $\epsilon > 0$
4. It assigns a non-negative mass to every rectangle in its domain: Let $A = \mathop{\times}_{j=1}^{d} (a_{j,0}, a_{j,1}]$. Then

$$\begin{aligned}\mathbb{V}_{\mathbf{C}}(A) &= \mathbb{P}_{\mathbf{C}}(\mathbf{U} \in A) \\ &= \sum_{l_1=0}^{1} \cdots \sum_{l_d=0}^{1} (-1)^{d+l_1+\cdots+l_d} \mathbf{C}(a_{1,l_1}, \ldots, a_{j,l_j}) > 0\end{aligned}$$

Proof for bivariate copulas

- ad 1.)

$$\mathbf{C}(u_1, 0) = \mathbb{P}(U_1 \leq u_1, U_2 \leq 0) = 0$$

- ad 2.)

$$\begin{aligned}\mathbf{C}(u_1, 1) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq 1) \\ &= \mathbb{P}(U_1 \leq u_1 | U_2 \leq 1) \cdot \mathbb{P}(U_2 \leq 1) \\ &= \mathbb{P}(U_1 \leq u_1) \cdot 1 \\ &= u_1\end{aligned}$$

Proof for bivariate copulas

- ad 3.)

$$\begin{aligned}\mathbf{C}(u_1, u_2) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) \\ &\leq \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2 + \epsilon) \\ &= \mathbf{C}(u_1, u_2 + \epsilon)\end{aligned}$$

- ad 4.)

$$A := [a_{1,0}, a_{1,1}] \times [a_{2,0}, a_{2,1}]$$

$$\begin{aligned}\mathbb{V}_{\mathbf{C}}(A) &= \mathbb{P}(\mathbf{U} \in A) \\ &= \mathbb{P}(a_{1,0} \leq U_1 \leq a_{1,1}, a_{2,0} \leq U_2 \leq a_{2,1})\end{aligned}$$

Convex combinations

Theorem

Let $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be copulas, and let* α *be any number in the unit interval* $[0, 1]$. *Then the weighted arithmetic mean*

$$\alpha \mathbf{C}_1 + (1 - \alpha) \mathbf{C}_2$$

is a copula, too.

Frechet bounds

- given perfect dependence $U_1 = U_2$:

$$\begin{aligned} \mathbf{C}(u_1, u_2) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) \\ &= \mathbb{P}(U_1 \leq u_1, U_1 \leq u_2) \\ &= \mathbb{P}(U_1 \leq \min(u_1, u_2)) \\ &= \min(u_1, u_2) \end{aligned}$$

- given perfect negative dependence $U_1 = 1 - U_2$:

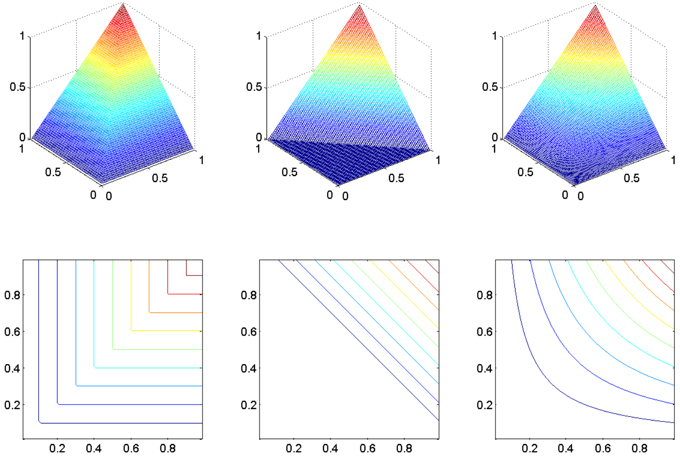
$$\begin{aligned} \mathbf{C}(u_1, u_2) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) \\ &= \mathbb{P}(U_1 \leq u_1, 1 - U_1 \leq u_2) \\ &= \mathbb{P}(U_1 \leq u_1, U_1 \geq 1 - u_2) \\ &= \mathbb{P}(1 - u_2 \leq U_1 \leq u_1) \\ &= \max\{u_1 - (1 - u_2), 0\} \\ &= \max\{u_1 + u_2 - 1, 0\} \end{aligned}$$

Independence copula

- given U_1 and U_2 independent of each other, it holds:

$$\begin{aligned} \mathbf{C}(u_1, u_2) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) \\ &= \mathbb{P}(U_1 \leq u_1) \cdot \mathbb{P}(U_2 \leq u_2) \end{aligned}$$

Frechet bounds



Sklar's Theorem

Let **F** denote a continuous d-dimensional distribution function with margins $F_1, \ldots, F_d$. Then:

- There exists a unique copula function **C** with

$$\mathbf{F}(x_1, \ldots, x_d) = \mathbf{C}\left(F_1\left(x_1\right), \ldots, F_d\left(x_d\right)\right)$$

  for all $x_1, \ldots, x_d \in \mathbb{R}$. This is equivalent to

$$\mathbf{F}\left(\mathbf{F}^{-1}\left(u_1\right), ..., \mathbf{F}^{-1}\left(u_d\right)\right) = \mathbf{C}\left(u_1, ..., u_d\right).$$

- For any copula **C** and any $x_1, \ldots, x_d \in \mathbb{R}$

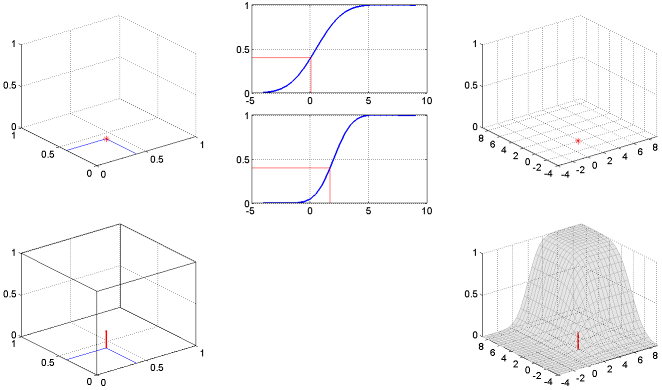
$$H_{\mathbf{C}} := \mathbf{C}(F_1(x_1), \ldots, F_d(x_d))$$

  is a joint distribution function with margins $F_1\left(x_1\right), \ldots F_d\left(x_d\right)$.

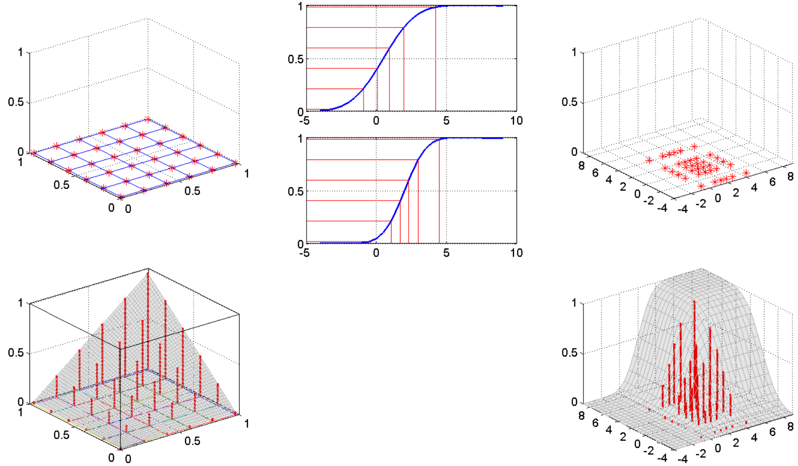
Sklar's Theorem: benefits

- given any multidimensional joint distribution function **F**, Sklar's theorem allows for a complete **extraction** of the **dependency structure** via

$$\mathbf{C}(u_1, \ldots, u_d) = \mathbf{F}(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$$



Extracting dependency



Canonical density representation

Theorem

Let $\mathbf{F}$ be a d-dimensional distribution function with density $\mathbf{f}$, and $\mathbf{C}$ be the copula of $\mathbf{F}$ according to Sklar's theorem, with copula density denoted by $\mathbf{c}$. Then $\mathbf{f}$ can be represented as

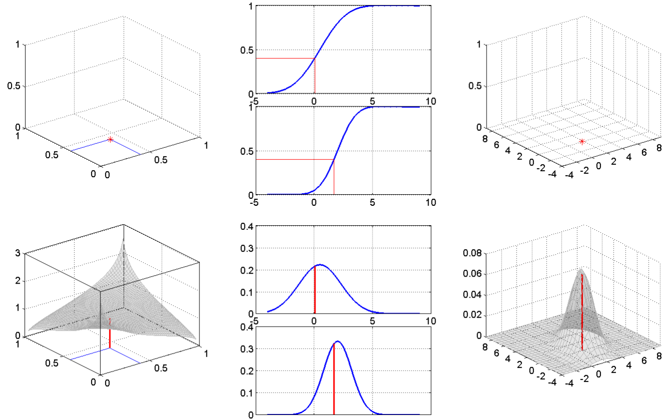
$$\mathbf{f}(x_1, \ldots, x_d) = \mathbf{c}(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdot \ldots \cdot f_d(x_d).$$

Hence the copula density can be calculated by

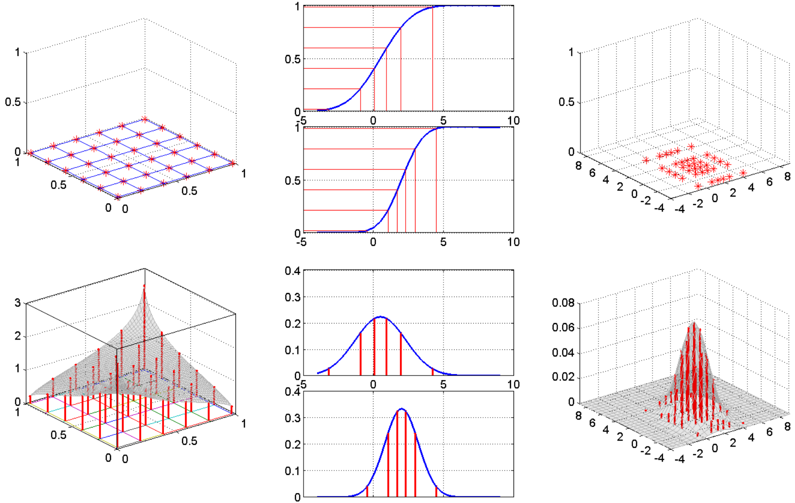
$$\mathbf{c}(u_1, \ldots, u_d) = \frac{\mathbf{f}(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \cdots f_d(F_d^{-1}(u_d))}.$$

Extracting copula density

$$c(u_1, u_2) = \frac{f\left(F^{-1}(u_1), F^{-1}(u_2)\right)}{f_1\left(F^{-1}(u_1)\right) \cdot f_2\left(F^{-1}(u_2)\right)}$$



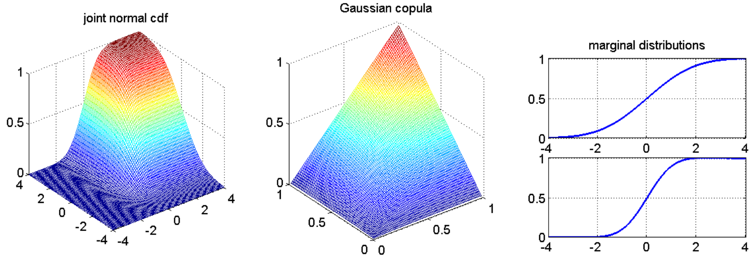
Extracting copula density



Gaussian copula

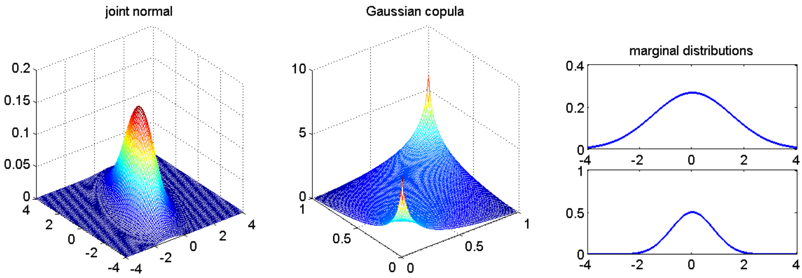
- decomposing the bivariate normal cdf $\Phi_{X,Y}$ into marginal cdfs Φ_X and Φ_Y and copula

$$\mathbf{C}^{Gau}(u_1, u_2) = \Phi_{X,Y}\left(\Phi_X^{-1}(u_1), \Phi_Y^{-1}(u_2)\right)$$

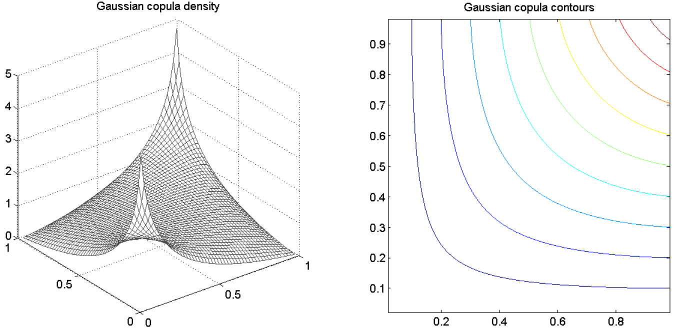


Gaussian copula density

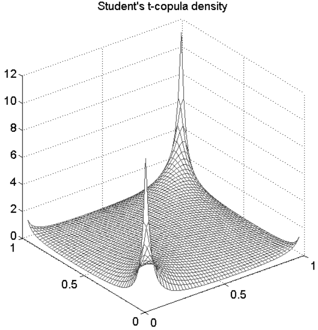
- decomposing the bivariate normal pdf

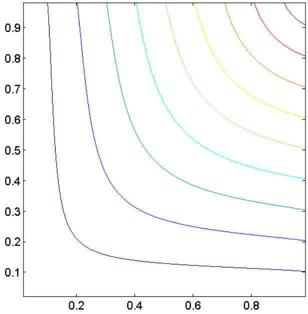


Implicit copulas: Gaussian copula



Implicit copulas: Student's t-copula





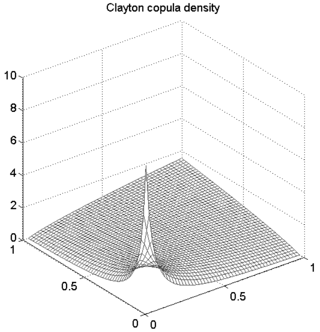
Sklar's Theorem: benefits

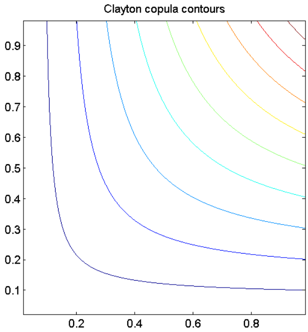
- more important direction of Sklar's theorem: it allows the **creation of new multivariate distribution functions**
- any d univariate marginal distributions can be linked together via the concept of copulas, yielding a valid d-dimensional distribution function
- substantially increasing the set of known parametric distributions
- margins and dependence structure can be modelled separately, allowing to benefit of existing univariate models

Parametric copula families

- Clayton copula family

$$\mathbf{C}\left(u_1, u_2\right) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}, \ \theta > 0$$

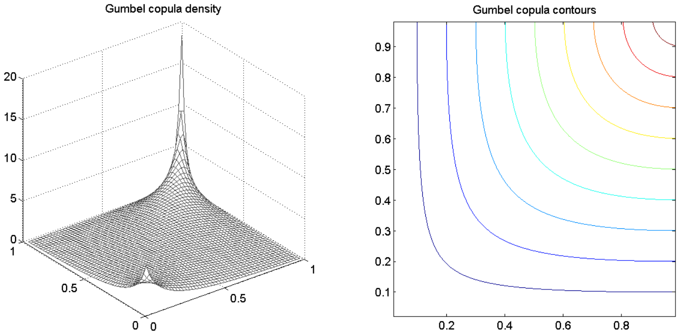




Parametric copula families

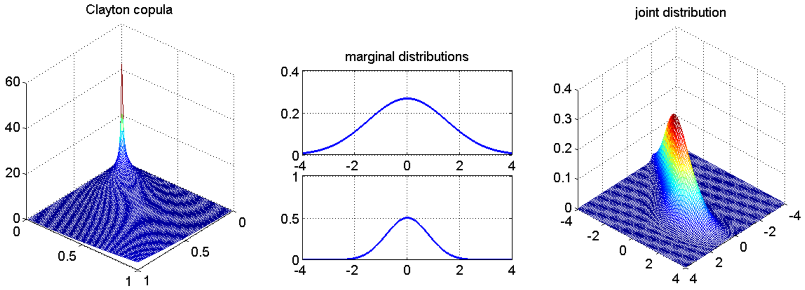
- Gumbel copula family

$$\mathbf{C}\left(u_1, u_2\right) = exp\left(-\left(\left(-\ln u_1\right)^{\theta} + \left(-\ln u_2\right)^{\theta}\right)^{1/\theta}\right), \ \theta \geq 1$$



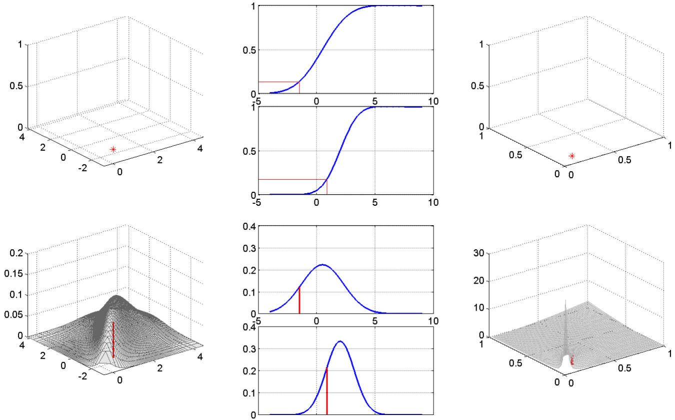
Creating joint distributions

- combining **any** copula with **any** continuous marginal distributions leads to a valid joint distribution



Creating joint distributions: calculation

$$f(x_1, x_2) = c(F(x_1), F(x_2)) \cdot f(x_1) \cdot f(x_2)$$



- for $\theta = 1$ the Gumbel copula family includes the independence copula:

$$
\begin{aligned}
\mathbf{C}(u_1, u_2) =& exp\left(-\left((-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right)^{1/\theta}\right) \\
\overset{\theta=1}{=}& exp\left(-\left((-\ln u_1)^{1} + (-\ln u_2)^{1}\right)^{1}\right) \\
=& exp\left(-\left(\ln u_1^{-1} + \ln u_2^{-1}\right)^{1}\right) \\
=& exp\left(-\ln\left(u_1^{-1} \cdot u_2^{-1}\right)\right) \\
=& exp\left(\ln\left(\left(u_1^{-1} \cdot u_2^{-1}\right)^{-1}\right)\right) \\
=& \left(u_1^{-1} \cdot u_2^{-1}\right)^{-1} = u_1 \cdot u_2
\end{aligned}
$$

- let F be a bivariate cumulative distribution function, with Clayton copula with parameter $\theta = 2$ and one exponentially distributed marginal distribution $F_1(x) = 1 - exp(-3x)$ and one standard normally distributed marginal distribution $F_2(x) \sim \mathcal{N}(0, 1)$
- calculate $F(2, 1)$:
  - transform point $(2, 1)$ to unit square with marginal cumulative distribution functions:

$$\begin{aligned} u_1 &= F_1(x_1) \\ &= 1 - exp(-3x_1) \\ &= 1 - exp(-6) \\ &= 0.9975 \end{aligned}$$

$$\begin{aligned} u_2 &= F_2(x_2) \\ &= \Phi(1) \\ &= 0.8413 \end{aligned}$$

- calculate copula value at (u_1, u_2):

$$\mathbf{C}(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$
$$= \left(0.9975^{-2} + 0.8413^{-2} - 1\right)^{-1/2}$$
$$= 0.8398$$

- calculation of Clayton copula density:
  - derivative with respect to u_1 :

$$\frac{\partial\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-1/\theta}}{\partial u_1}=\left(-\frac{1}{\theta}\right)\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}-1}\cdot\left(-\theta u_1^{-\theta-1}\right)$$
$$=\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}-1}\cdot\left(u_1^{-\theta-1}\right)$$

  - derivative with respect to u_2 :

$$\frac{\partial\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}-1}\cdot\left(u_1^{-\theta-1}\right)}{\partial u_2}=$$

$$=\left(u_1^{-\theta-1}\right)\left(-\frac{1}{\theta}-1\right)\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}-2}\left(-\theta u_2^{-\theta-1}\right)$$
$$=(1+\theta)\left(u_1u_2\right)^{-\theta-1}\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}-2}$$

- evaluate copula density at (u_1, u_2):

$$\begin{aligned}\mathbf{c}\left(u_1, u_2\right) &= (1+\theta)\left(u_1 u_2\right)^{-\theta-1}\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}-2} \\ &= 3\,(0.9975 \cdot 0.8413)^{-3}\left(0.9975^{-2}+0.8413^{-2}-1\right)^{-2.5} \\ &= 2.1205\end{aligned}$$

- evaluate marginal pdfs

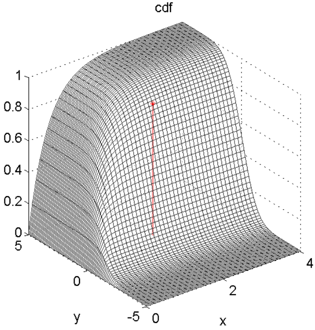
$$\begin{aligned}f_1\left(x_1\right) &= \lambda exp\left(-\lambda x_1\right) \\ &= 3 exp\left(-3 \cdot 2\right) \\ &= 0.0074\end{aligned}$$

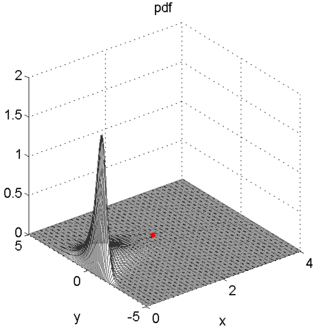
$$\begin{aligned}f_2\left(x_2\right) &= \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{x_2^2}{2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{1}{2}\right) \\ &= 0.2420\end{aligned}$$

$$
\begin{aligned}
\mathbf{f}(x_1, \ldots, x_d) &= \mathbf{c}(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdot \ldots \cdot f_d(x_d) \\
&= 2.1205 \cdot 0.0074 \cdot 0.2420 \\
&= 0.0038
\end{aligned}
$$

$$
\begin{aligned}
F(x_1, x_2) &= \mathbf{C}\left(F_1(x_1), F_2(x_2)\right) \\
&= \mathbf{C}\left((1 - exp(-\lambda x_1)), \Phi(x_2)\right) \\
&= \left((1 - exp(-\lambda x_1))^{-\theta} + (\Phi(x_2))^{-\theta} - 1\right)^{-\frac{1}{\theta}}
\end{aligned}
$$

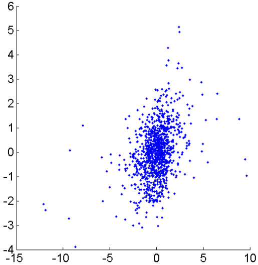
Joint density





Main benefit

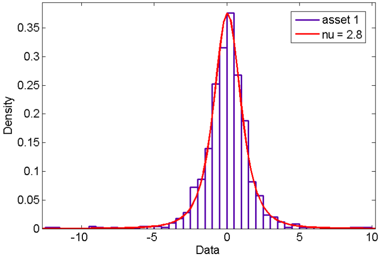
- the figure below shows a scatterplot of a given dataset, together with a scatterplot of the dataset transformed on the unit square

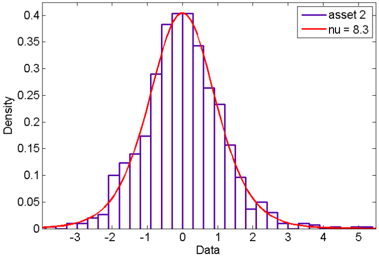


- as can be seen, the dependence structure does not entail any exceptional features: it is the same as implicitly used by the normal distribution

Main benefit

- also, the marginal distributions do not exhibit any unusual features: both are distributed according to a Student's t-distribution





Main benefit

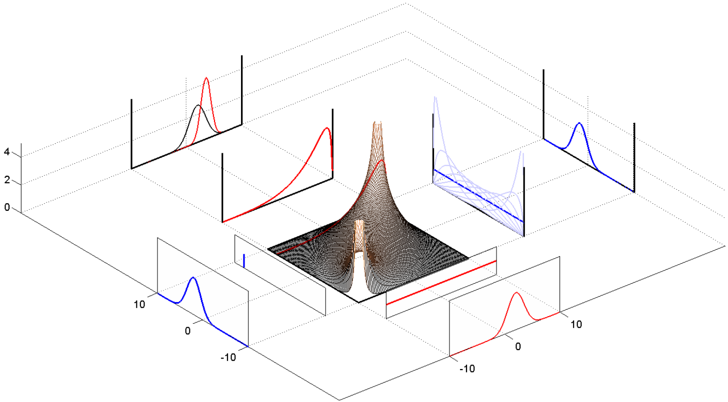
- however, although both the marginal distributions and the underlying dependence structure follow well-known and often seen parametric families, without copulas **there is no way to simultaneously model all components with the required parametric distribution**
- in order to simultaneously model margins distributed according to a Student's t-distribution, we have to use the two-dimensional Student's t-distribution, which forces both marginal distributions to have **the same degrees-of-freedom parameter** ν
- hence, instead of modelling margin 1 with $\nu_1 = 2.8$ and margin 2 with 8.6, one has to make a compromise and model both margins with some value $\nu_1 < \nu < \nu_2$
- moreover, the implicit dependence structure of the two-dimensional Student's t-distribution would not match the given Gaussian dependence structure in this case

Main benefit

- even without leaving the world of familiar and often used parametric distributions, the **copula framework** is able to break up the existing restrictions in multi-dimensional modelling
- moreover, copulas allow new and probably more exceptional dependence structures than the ones implicitly applied in multivariate normal or Student's *t*-distributions
- $\rightarrow$Matlab application *I*: create bivariate distribution

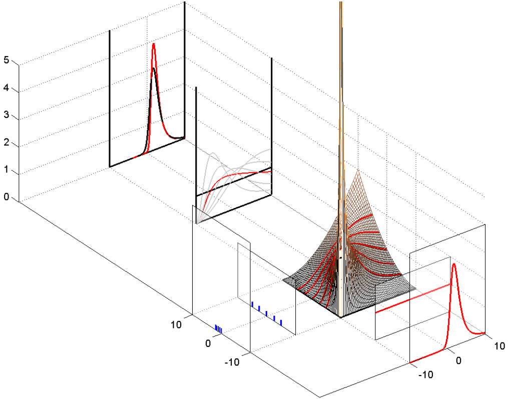
Example: bivariate distribution

- map the given value to “quantile world”, get conditional distribution of “quantiles” of second component, and map the result back to absolute terms



Information updating

- additional information about first component can be used to update second component



Simulating from copulas

- the core of any simulation algorithm is the formula for Monte Carlo simulation of a random variable with given distribution F_X :

$$F_X^{-1}(U) \sim F_X$$

- extension to multi-variate case given by conditional inversion method valid for arbitrary continuous distributions (not only copulas):
  - simulate independent values $(w_1, w_2, ..., w_n)$, each from uniform distribution
  - set

$$\begin{aligned} u_1 &= w_1 \\ u_2 &= F_{2|1}^{-1}(w_2|u_1) \\ u_3 &= F_{3|12}^{-1}(w_3|u_1, u_2) \\ &\vdots \\ u_n &= F_{n|1\ldots(n-1)}^{-1}(w_n|u_1, \ldots, u_{n-1}) \end{aligned}$$

Conditional probabilities via copula

Theorem

Let $\mathbf{C}$ *be the 2-dimensional copula of random vector* $\mathbf{U} = (U_1, U_2)$ *. Then the probability of one arbitrary component* i *being smaller than some specified value, conditional on the realizations of the remaining component* j *is*

$$\mathbb{P}(U_i \leq u_i | U_j = u_j) = \frac{\partial \mathbf{C}(u_1, u_2)}{\partial u_j}.$$

Explanation

- given that we already know the realization of U_1, and we know that the relationship between U_1 and U_2 is given by copula $\mathbf{C}$, we can update our knowledge about U_2
- the updated distribution function $C_{2|1}(x|u_1)$ of U_2 can be calculated by derivation of copula $\mathbf{C}$ with respect to the already known component:

$$C_{2|1}(x|u_1) = \mathbb{P}(U_2 \leq x | U_1 = u_1) = \frac{\partial \mathbf{C}(u_1, x)}{\partial u_1}$$

Example: Clayton copula

- calculation of Clayton copula v-function:
  $v(u_2, u_1) = \mathbb{P}(U_2 \leq u_2 | U_1 = u_1)$
- derivative with respect to u_1 :

$$\frac{\partial \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}}{\partial u_1} = \left(-\frac{1}{\theta}\right)\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}-1} \cdot \left(-\theta u_1^{-\theta-1}\right)$$

$$= \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}-1} \cdot \left(u_1^{-\theta-1}\right)$$

$$\Rightarrow v(u_2, u_1) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}-1} \cdot \left(u_1^{-\theta-1}\right)$$

Inverse v-function

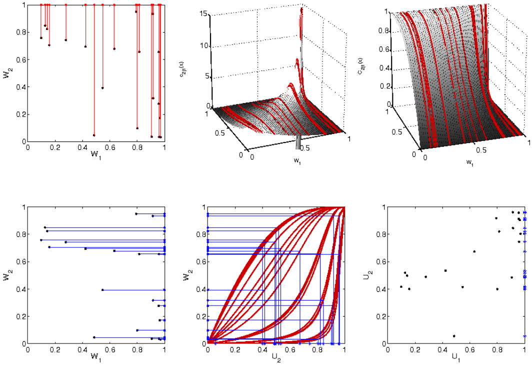
- as simulation involves application of inverse cumulative distribution function, inverse function is required also for conditional case

$$u_2 = \left(u_1^{-\theta} + y^{-\theta} - 1\right)^{-\frac{1}{\theta}-1} \cdot \left(u_1^{-\theta-1}\right)$$

$$\left(\frac{u_2}{\left(u_1^{-\theta-1}\right)}\right)^{\left(\frac{\theta}{1+\theta}\right)} = \left(u_1^{-\theta} + y^{-\theta} - 1\right)$$

$$v^{-1}(u_2, u_1) = y = \left(\left(\frac{u_2}{\left(u_1^{-\theta-1}\right)}\right)^{\left(\frac{\theta}{1+\theta}\right)} + 1 - u_1^{-\theta}\right)^{-1/\theta}$$

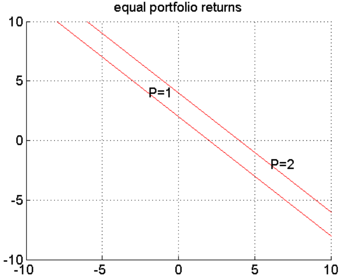
v-function



- →Matlab application *II*: simulation via copula

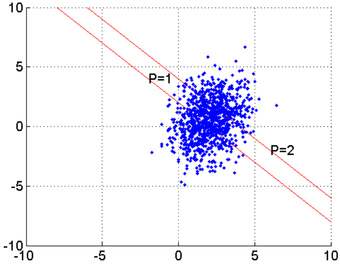
Portfolio returns: linear case

- portfolio return, discrete case: $r_P = wr_1 + (1 - w)\, r_2$
- formula also can be applied as approximation for log-returns
- given equal portfolio weights for individual assets, asset return realizations with equal associated portfolio returns are lying on diagonal line



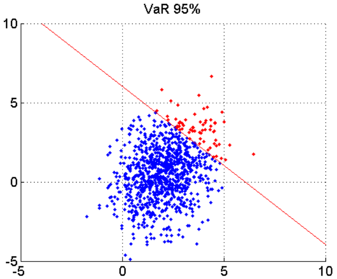
Example

- return pairs with associated portfolio return between 1 and 2



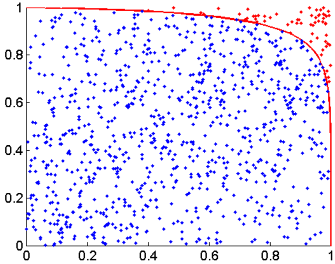
Determining portfolio VaR

- estimating empirical portfolio-VaR_α amounts to dragging down the transverse line until $(1-\alpha)$ −percent of points of loss pairs (negative returns) are above the line



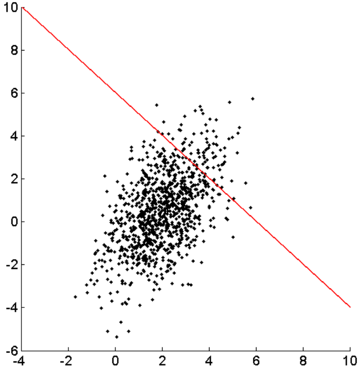
Mapping to unit square

- after mapping with marginal distributions the straight line gets bent

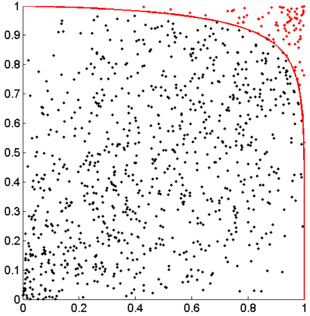


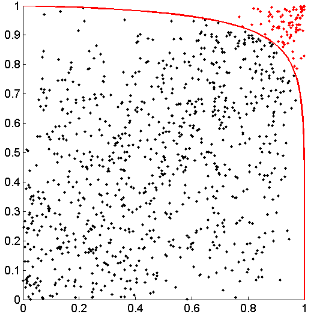
Model risk

- adapting the portfolio VaR_α to the case of a Gaussian copula leads to underestimated risk for the case of an actually underlying copula with high dependence for large losses: for example, exceedance frequency of 8-percent for $VaR_{0.95}$



Comparison on unit square





Tail dependence

Definition

If for a bivariate copula C

$$\lim_{u \to 0^+} \mathbb{P}(U_1 < u | U_2 < u) = \lim_{u \to 0^+} \frac{C(u,u)}{u} = \lambda_l,$$

with $\lambda_l \in (0,1]$, then C has lower tail dependence, and if

$$\lim_{u \to 1^-} \mathbb{P}(U_1 > u | U_2 > u) = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u} = \lambda_u,$$

with $\lambda_u \in (0,1]$, then C has upper tail dependence.

- probability of being extreme, given that the other asset is extreme, too

Copula, causality and information

- copula model does not provide information about the forcing variable: direction of causality remains outside of copula model
- information transmission in both directions
    - through causality knowledge of realization of X_1 enables inference about X_2
    - incorporation of probabilistic perspective on X_1 allows inference from X_2 to X_1 via Bayes theorem (in contrast to regression model, where knowledge of Y does not allow for backward reasoning about X)
- without intention to intervene in the underlying model, distinguishing between $X_1 \rightarrow X_2$, $X_1 \leftarrow X_2$ or $X_1 \leftrightarrow X_2$ may be unnecessary: information updating is possible anyway