Riskmanagement Exercises

Groll

Seminar für Finanzökonometrie

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Single credit, default-only

- considering a **single credit**, with **losses only** possible **in** event of **default**
- depending on the outcome of the investment project, **different recovery rates** are possible in the event of default

Exercise

A firm has to repay its debt of 100€ in one year. In addition to the capital reserves of 30€ , the firm will earn some money from an investment it has made. However, the exact cash flow associated with the investment project is uncertain, and can be described by the following distribution:

Probability	0.9	0.04	0.03	0.03
Cash flow	110	50	30	15

Derive the associated **loss distribution** from the perspective of the lending bank, and calculate the **probability of default**, the **expected loss**, the **loss given default** and the **value-at-risk** at confidence levels 95% and 99%.

Single Credit

Probability	0.9	0.04	0.03	0.03
Project CF	110	50	30	15
Asset values	140	80	60	45
Losses	$(100-140)^+$	$(100-80)^+$	$(100-60)^+$	$(100-45)^+$
Losses	0	20	40	55
Cumulated	0.9	0.94	0.97	1

Quantitative characteristics

- **probability of default**: the default event entails each outcome with incomplete repayment of debt:

$$PD = 0.04 + 0.03 + 0.03 = 0.1$$

- **expected loss**:

$$\mathbb{E}[L] = 0.9 \cdot 0 + 0.04 \cdot 20 + 0.03 \cdot 40 + 0.03 \cdot 55$$

Quantitative characteristics

- **loss given default**: probabilities have to be scaled up

$$
\begin{aligned}
LGD &= \frac{\mathbb{P}(L = 20)}{PD} \cdot 20 + \frac{\mathbb{P}(L = 40)}{PD} \cdot 40 + \frac{\mathbb{P}(L = 55)}{PD} \cdot 55 \\
&= \frac{0.04}{0.1} \cdot 20 + \frac{0.03}{0.1} \cdot 40 + \frac{0.03}{0.1} \cdot 55 \\
&= 36.5
\end{aligned}
$$

- **value-at-risk**:

$$
VaR_{0.95} = 40, \quad VaR_{0.99} = 55
$$

Downgrade risk

- even without actual default, the value of a credit or bond can decrease because of decreasing credit quality
- the loss would have to be realized in case that the debt is resold at the capital market

Exercise

A bank has provided a loan to a company. In return, the company has to make interest payments of 12€ each year, additional to the repayment of the loan with nominal amount 100€ in four years. Hence, outstanding payments are given by

time	1	2	3	4	5
cash flow	12	12	12	112	0

Calculate for each of both internal rating categories of the bank the value of the loan in one year, when the first interest payment in $t = 1$ is excluded from the calculation. Forward yield curves for each rating category are given on the next slide.

Yield curves

- to revaluate the bond at the end of the year, future cash flows have to be discounted with the interest rates of the yield curves prevailing at that time
- here: revaluation assumes that the interest payment in $t = 1$ already has occured: it does not have to be considered in the revaluation formula
- forward yield curves: interest rates expected to prevail one year ahead

Duration	1	2	3	4	5
A	2.0	2.62	2.82	3.26	3.84
B	8.2	9.4	10.2	11.6	12.4

revaluation

- in case of future credit quality A:

$$V^A = \frac{12}{1.02} + \frac{12}{1.0262^2} + \frac{112}{1.0282^3}$$
$$= 126.19$$

- in case of future credit quality B:

$$V^B = \frac{12}{1.082} + \frac{12}{1.094^2} + \frac{112}{1.102^3}$$
$$= 104.8$$

Downgrade and default risk together

rating transitions		end of year A	end of year B	end of year default
beginning of the year	A	0.9	0.05	0.05
	B	0.05	0.75	0.2

- residual values, given that default occurs

residual value	40	55	65	80
probability	0.2	0.4	0.3	0.1

Loss distribution

- loss distribution, given that value today is 110, and firm is rated B at present
- present rating category determines probabilities

	A	B	default			
value	126	104	80	65	55	40
loss	$(110-126)^+$	$(110-104)^+$	30	45	55	70
loss	0	6	30	45	55	70
$\mathbb{P}\|B$	0.05	0.75	$0.1 \cdot 0.2$	$0.3 \cdot 0.2$	$0.4 \cdot 0.2$	$0.2 \cdot 0.2$
$\mathbb{P}\|B$	0.05	0.75	0.02	0.06	0.08	0.04
cumul.	0.05	0.8	0.82	0.88	0.96	1

- reading off the value-at-risk: $VaR_{0.95} = 55$

Credit portfolio

- combination of one A-rated bond B_1 and one B-rated bond B_2
- no distinction in case of default: information about recovery rates left unconsidered
- marginal distributions are given according to transition matrix

rating transitions		end of year A	end of year B	end of year default
beginning of the year	A	0.9	0.05	0.05
beginning of the year	B	0.05	0.75	0.2

Joint probabilities

- joint probabilities have to be calculated
- in order to come up with joint probabilities, we have to model the dependence structure of the underlying asset processes
- hence, we have to find an appropriate copula family, and determine the copula parameter based on an estimated correlation coefficients for the underlying assets

		B_1			
		default	B	A	
B_2	A				0.05
	B				0.75
	default				0.2
		0.05	0.05	0.9	

Calculation of factor models

- idea: derive asset value correlation from factor models
- asset value correlation in turn has to be converted into an appropriate copula parameter
- representation with factors

$$r_1 = 0.6X_1 + 0.4X_2 + b_1\epsilon_1$$

$$r_2 = 0.2X_1 + 0.8X_3 + b_2\epsilon_2$$

- fractions of variance explained by factors shall be 0.6 in case of bond B_1 and 0.4 in case of bond B_2

Required input parameters

- parameters associated with factors are given by

			correlation		
		volatility	X_1	X_2	X_3
index	X_1	1.2	1	0.2	0.4
	X_2	1.4		1	0.2
	X_3	1.4			1

Calculation bond B_1

- calculation of factor variance

$$\begin{aligned}\mathbb{V}\left(0.6X_1+0.4X_2\right)=&0.6^2\mathbb{V}\left(X_1\right)+0.4^2\mathbb{V}\left(X_2\right)\\&+2Cov\left(0.6X_1,0.4X_2\right)\\=&0.6^2\cdot 1.2^2+0.4^2\cdot 1.4^2\\&+2\cdot 0.6\cdot 0.4\cdot\rho_{X_1X_2}\cdot\sigma_{X_1}\sigma_{X_2}\\=&0.993\end{aligned}$$

- overall variance, according to information about idiosyncratic influence:

$$\begin{aligned}\mathbb{V}\left(r_1\right)&=\frac{\mathbb{V}\left(0.6X_1+0.4X_2\right)}{0.6}\\&=\frac{0.993}{0.6}=1.655\end{aligned}$$

Calculation bond B_2

- calculation of factor variance

$$
\begin{aligned}
\mathbb{V}\left(0.2X_1+0.8X_3\right)=&0.2^2\mathbb{V}\left(X_1\right)+0.8^2\mathbb{V}\left(X_3\right)\\
&+2Cov\left(0.2X_1,0.8X_3\right)\\
=&0.2^2\cdot 1.2^2+0.8^2\cdot 1.4^2\\
&+2\cdot 0.2\cdot 0.8\cdot \rho_{X_1X_3}\cdot \sigma_{X_1}\sigma_{X_3}\\
=&1.312+0.32\cdot 0.4\cdot 1.2\cdot 1.4\\
=&1.527
\end{aligned}
$$

- overall variance, according to information about idiosyncratic influence:

$$
\begin{aligned}
\mathbb{V}\left(r_2\right)&=\frac{\mathbb{V}\left(0.2X_1+0.8X_3\right)}{0.4}\\
&=\frac{1.527}{0.4}=3.8175
\end{aligned}
$$

Calculation of covariance

- given the factor models for the underlying asset returns of the two bonds, we can calculate the covariance between the underlying asset returns

$$
\begin{aligned}
Cov\left(r_1, r_2\right) =& Cov\left(0.6X_1 + 0.4X_2, 0.2X_1 + 0.8X_3\right) \\
=& 0.6 \cdot 0.2 \cdot Cov\left(X_1, X_1\right) + 0.6 \cdot 0.8 \cdot Cov\left(X_1, X_3\right) \\
& + 0.4 \cdot 0.2 \cdot Cov\left(X_2, X_1\right) + 0.4 \cdot 0.8 \cdot Cov\left(X_2, X_3\right) \\
=& 0.12\mathbb{V}\left(X_1\right) + 0.48 \cdot \rho_{X_1 X_3}\sigma_{X_1}\sigma_{X_3} \\
& + 0.08\rho_{X_2 X_1}\sigma_{X_2}\sigma_{X_1} + 0.32\rho_{X_2 X_3}\sigma_{X_2}\sigma_{X_3} \\
=& 0.173 + 0.48 \cdot 0.672 + 0.08 \cdot 0.336 + 0.32 \cdot 0.392 \\
=& 0.648
\end{aligned}
$$

Calculation of correlation

- hence, the asset correlation is given by

$$\rho_{r_1 r_2} = \frac{Cov\left(r_1, r_2\right)}{\sigma_{r_1}\sigma_{r_2}} = \frac{0.648}{\sqrt{\mathbb{V}\left(r_1\right)}\sqrt{\mathbb{V}\left(r_2\right)}} = \frac{0.648}{\sqrt{1.655}\sqrt{3.8175}} = 0.258$$

Joint probabilities

- assumption: underlying asset processes follow a joint distribution with dependence structure given by Clayton copula C^{Clay} :

$$C^{Clay}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

- in order to achieve an asset correlation of 0.258 with standard normally distributed margins, the Clayton copula parameter has to be approximately equal to 0.45

Joint default probability

- the joint default probability is given by

$$\begin{aligned}\mathbb{P}\left(B_1 = default, B_2 = default\right) &= \mathbb{P}\left(A_1 \leq 0.05, A_2 \leq 0.2\right)\\ &= C^{Clay}\left(0.05, 0.2\right)\\ &= \left(0.05^{-0.45} + 0.2^{-0.45} - 1\right)^{-1/0.45}\\ &= 0.029\end{aligned}$$

Joint probabilities

- table with cumulative margins

		B_1 default	B_1 B	B_1 A	cumulated
B_2	A				1
	B				0.95
	default				0.2
cumulated		0.05	0.1	1	

Joint probabilities

$$\begin{aligned}
\mathbb{P}\left(B_1 = default, B_2 = B\right) &= \mathbb{P}\left(A_1 \leq 0.05, 0.2 < A_2 \leq 0.95\right) \\
&= C^{Clay}\left(0.05, 0.95\right) - C^{Clay}\left(0.05, 0.2\right) \\
&= 0.049 - 0.029 \\
&= 0.02
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left(B_1 = B, B_2 = default\right) &= \mathbb{P}\left(0.05 < A_1 \leq 0.1, A_2 \leq 0.2\right) \\
&= C^{Clay}\left(0.1, 0.2\right) - C^{Clay}\left(0.05, 0.2\right) \\
&= 0.049 - 0.029 \\
&= 0.02
\end{aligned}$$

Joint probabilities

$$
\begin{aligned}
\mathbb{P}(B_1 = B, B_2 = B) =& \mathbb{P}(0.05 < A_1 \leq 0.1, 0.2 < A_2 \leq 0.95) \\
=& \mathbb{P}(A_1 \leq 0.1, A_2 \leq 0.95) - \mathbb{P}(A_1 \leq 0.05, A_2 \leq 0.2) \\
& - \mathbb{P}(0.05 < A_1 \leq 0.1, A_2 \leq 0.2) \\
& - \mathbb{P}(A_1 \leq 0.05, 0.2 < A_2 \leq 0.95) \\
=& C^{Clay}(0.1, 0.95) - 0.029 - 0.02 - 0.02 \\
=& 0.098 - 0.069 \\
=& 0.029
\end{aligned}
$$

Joint probabilities

		B_1			
		default	B	A	
B_2	A	0.001	0.001	0.048	0.05
	B	0.02	0.029	0.701	0.75
	default	0.029	0.02	0.151	0.2
		0.05	0.05	0.9	

Default correlation

- the default correlation is given by

$$\varrho_{B_1 B_2} = \frac{\mathbb{P}\left(B_1 = \textit{default}, B_2 = \textit{default}\right) - PD_1 \cdot PD_2}{\sqrt{PD_1\left(1 - PD_1\right)}\sqrt{PD_2\left(1 - PD_2\right)}}$$

$$= \frac{0.029 - 0.05 \cdot 0.2}{\sqrt{0.05 \cdot 0.95}\sqrt{0.2 \cdot 0.8}}$$

$$= 0.218$$

Exercise

The two-dimensional distribution function H shall be given as a composition of a Frank copula C^{Fra} with parameter $\alpha = 1$ and Pareto distributed marginal distributions. While the parameters of margin 1 are given by $x_{min,1} = 0.1$ and $k = 2$, the minimal possible value $x_{min,2}$ of the second margin shall be given by $x_{min,2} = 0.2$. Compute the value of the cumulative distribution function H as well as the value of the associated probability distribution function h at the point $(0.4, 0.4)$. Furthermore, calculate the probability that margin 1 exceeds the value $c_1 = 0.4$ while margin 2 simultaneously exceeds the value $c_2 = 0.4$.

Distributions

The Frank copula is given by

$$C^{Fra}(u,v) = -\frac{1}{\alpha}\ln\left(1 + \frac{(exp(-\alpha u)-1)(exp(-\alpha v)-1)}{exp(-\alpha)-1}\right)$$

with copula density function

$$c^{Fra}(u,v) = \frac{\alpha(1-exp(-\alpha))\,exp(-\alpha(u+v))}{((1-exp(-\alpha))-(1-exp(-\alpha u))(1-exp(-\alpha v)))^2}.$$

The Pareto distribution is given by

$$F(x) = 1-\left(\frac{x_{min}}{x}\right)^k$$

with quantile function

$$F^{-1}(x) = \frac{x_{min}}{\sqrt[k]{1-x}}$$

and probability density function

$$f(x) = \left(\frac{x_{min}}{x}\right)^k\left(\frac{k}{x}\right).$$

Transformation to quantiles

- transforming the point of interest into quantile coordinates by application of the marginal cumulative distribution functions:

$$
\begin{aligned}
F_1(0.4) &= 1 - \left(\frac{x_{min,1}}{0.4}\right)^k \\
&= 1 - \left(\frac{0.1}{0.4}\right)^2 \\
&= 0.938
\end{aligned}
$$

$$
\begin{aligned}
F_2(0.4) &= 1 - \left(\frac{x_{min,2}}{0.4}\right)^k \\
&= 1 - \left(\frac{0.2}{0.4}\right)^2 \\
&= 0.75
\end{aligned}
$$

Evaluating copula function

- insertion into copula function:

$$\begin{aligned} C^{Fra}(0.938, 0.75) &= -\frac{1}{\alpha} \ln\left(1 + \frac{(exp(-\alpha u) - 1)(exp(-\alpha v) - 1)}{exp(-\alpha) - 1}\right) \\ &= -\ln\left(1 + \frac{(exp(-0.938) - 1)(exp(-0.75) - 1)}{exp(-1) - 1}\right) \\ &= -\ln\left(1 + \frac{(-0.609)(-0.528)}{-0.632}\right) \\ &= 0.711 \end{aligned}$$

- combining the results leads to

$$H(0.4, 0.4) = C^{Fra}(0.938, 0.75) = 0.711$$

Marginal densities

- evaluation of marginal densities:

$$f_1\left(0.4\right) = \left(\frac{x_{min,1}}{x}\right)^k \left(\frac{k}{x}\right) = \left(\frac{0.1}{0.4}\right)^2 \left(\frac{2}{0.4}\right) = 0.313$$

$$f_2\left(0.4\right) = \left(\frac{x_{min,2}}{x}\right)^k \left(\frac{k}{x}\right) = \left(\frac{0.2}{0.4}\right)^2 \left(\frac{2}{0.4}\right) = 1.25$$

Copula density

- evaluation of copula density at point in quantile coordinates:

$$
\begin{aligned}
c^{Fra}(u,v) &= c^{Fra}(0.938, 0.75) \\
&= \frac{\alpha\left(1-exp\left(-\alpha\right)\right)exp\left(-\alpha\left(u+v\right)\right)}{\left(\left(1-exp\left(-\alpha\right)\right)-\left(1-exp\left(-\alpha u\right)\right)\left(1-exp\left(-\alpha v\right)\right)\right)^2} \\
&= \frac{\left(1-exp\left(-1\right)\right)exp\left(-\left(u+v\right)\right)}{\left(\left(1-exp\left(-1\right)\right)-\left(1-exp\left(-u\right)\right)\left(1-exp\left(-v\right)\right)\right)^2} \\
&= \frac{0.632\cdot exp\left(-\left(0.938+0.75\right)\right)}{\left(0.632-\left(1-exp\left(-0.938\right)\right)\left(1-exp\left(-0.75\right)\right)\right)^2} \\
&= \frac{0.117}{\left(0.632-3.21\right)^2} \\
&= 1.211.
\end{aligned}
$$

Copula density

- combining the components leads to:

$$h(0.4, 0.4) = c^{Fra}(0.938, 0.75) \cdot f_1(0.4) \cdot f_2(0.4)$$
$$= 1.211 \cdot 0.313 \cdot 1.25$$
$$= 0.474$$

Exceedance probabilities

- converting the exceedance values into quantile information:

$$
\begin{aligned}
u_1 &= F_1(c_1) \\
&= 1 - \left(\frac{x_{min,1}}{0.4}\right)^k \\
&= 1 - \left(\frac{0.1}{0.4}\right)^2 \\
&= 0.938
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= F_2(c_2) \\
&= 1 - \left(\frac{x_{min,2}}{0.4}\right)^k \\
&= 1 - \left(\frac{0.2}{0.4}\right)^2 \\
&= 0.75
\end{aligned}
$$

Joint exceedance probability

- the joint exceedance probability is the copula h-volume of $[0.938, 1] \times [0.75, 1]$
- the h-volume can be calculated as

$$\begin{aligned} p &= C(1,1) + C(0.938, 0.75) - C(1, 0.75) - C(0.938, 1) \\ &= 1 + C(0.938, 0.75) - 0.75 - 0.938 \\ &= 0.711 - 0.688 \\ &= 0.023 \end{aligned}$$

- hence, the exceedance probability is 2.1%