Slides for Risk Management

Recap: VaR and ES

Groll

Seminar für Finanzökonometrie

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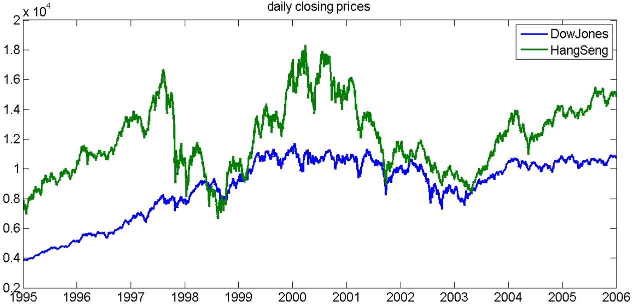
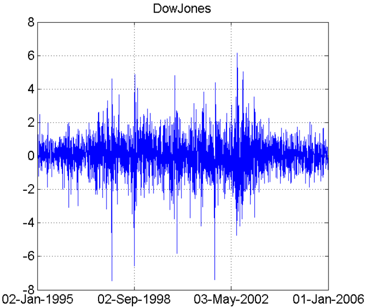
Price chart

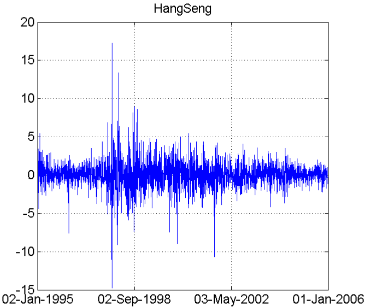
- daily closing prices of Dow Jones and Hang Seng index in estimation set



Percentage log-returns

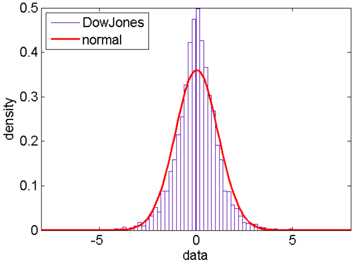
- **stylized facts:** volatility clustering, fat tailedness





Fitting normal distribution

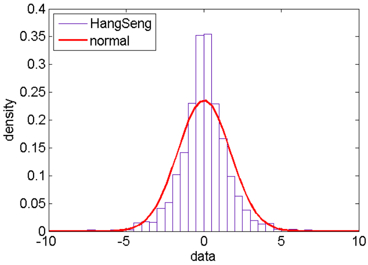
- ML estimation for normal distribution leads to estimated parameter values $\hat{\mu}_1 = 0.039$ and $\hat{\sigma}_1 = 1.107$ for percentage log-returns of Dow Jones
- empirical data: leptokurtosis visible, skewness not



Fitting normal distribution

- ML estimation for normal distribution leads to estimated parameter values $\hat{\mu}_2 = 0.024$ and $\hat{\sigma}_2 = 1.695$ for percentage returns of Hang Seng
- estimated covariance between both: $\widehat{Cov} = 0.309$
- calculate estimated **correlation**:

$$\hat{\rho} = \frac{\widehat{Cov}}{\hat{\sigma}_1 \hat{\sigma}_2} = \frac{0.309}{1.107 \cdot 1.695} = 0.165$$



Calculating risk measures

- transform returns to losses: $\mu_L = -\mu$
- quantiles normal distribution

$$\Phi^{-1}(0.95) = 1.645$$

$$\Phi^{-1}(0.99) = 2.326$$

$$\Phi^{-1}(0.999) = 3.090$$

- formulas for risk measures under normal distribution

$$VaR_\alpha = \mu_L + \sigma\Phi^{-1}(\alpha)$$

$$ES_\alpha = \mu_L + \sigma\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}$$

Calculation

- examplary calculation VaR :

$$\begin{aligned} VaR_{(1),0.95} &= -\mu_1 + \sigma_1 \Phi^{-1}(0.95) \\ &= -0.039 + 1.107 \cdot \Phi^{-1}(0.95) \\ &= -0.039 + 1.107 \cdot 1.645 \\ &= 1.782 \end{aligned}$$

- control results:

$$VaR_{(1),0.95} = 1.781, \; VaR_{(1),0.99} = 2.535, \; VaR_{(1),0.999} = 3.381$$

$$VaR_{(2),0.95} = 2.764, \; VaR_{(2),0.99} = 3.919, \; VaR_{(2),0.999} = 5.214$$

Expected shortfall

- normal distribution pdf:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)$$

- standard normal distribution pdf:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{x^2}{2}\right)$$

- examplary calculation:

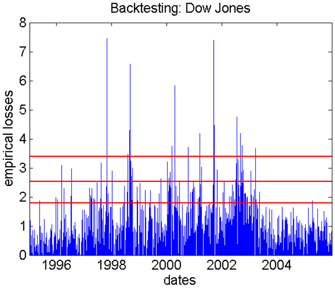
$$\begin{aligned} ES_{(1),0.99} &= -\mu_1 + \sigma_1 \frac{\phi\left(\Phi^{-1}(0.99)\right)}{1-0.99} \\ &= -0.039 + 1.107 \cdot \frac{\phi(2.326)}{0.01} \\ &= -0.039 + 1.107 \cdot \frac{0.0267}{0.01} \\ &= 2.911 \end{aligned}$$

Expected shortfall

- control results expected shortfall
  - $ES_{(1),0.95} = 2.244, \quad ES_{(1),0.99} = 2.911, \quad ES_{(1),0.999} = 3.687$
  - $ES_{(2),0.95} = 3.473, \quad ES_{(2),0.99} = 4.494, \quad ES_{(2),0.999} = 5.684$

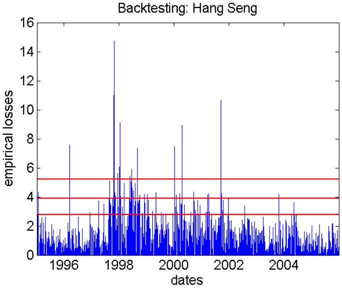
Backtesting

target	0.05	0.01	0.001
exceedances	0.044	0.015	0.005



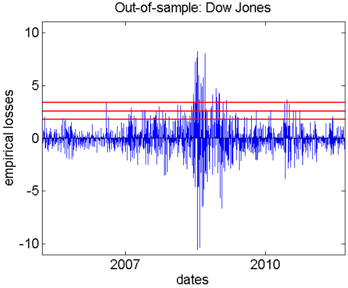
Backtesting

target	0.05	0.01	0.001
exceedances	0.043	0.016	0.006



Out-of-sample

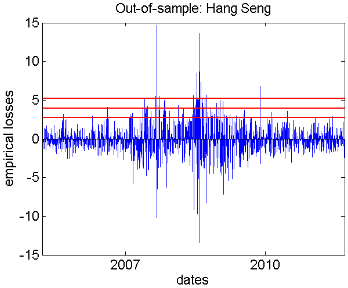
target	0.05	0.01	0.001
exceedances	0.075	0.036	0.018



- note: the exceedance frequency for $\alpha = 0.999$ is 18 times higher than wanted

Out-of-sample

target	0.05	0.01	0.001
exceedances	0.059	0.027	0.011



Explanation

- bad performance could be due to wrong model assumptions:
  - **normality**: empirical returns have fat tails, skewness
  - **no time-variation**: modelling with static distribution assumed to remain constant during all times

Multi-period moments with independency

- **expectation:** (independence unnecessary)

$$\mathbb{E}\left[r_{t,t+n-1}^{log}\right] = \mathbb{E}\left[\sum_{i=0}^{n-1} r_{t+i}^{log}\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[r_{t+i}^{log}\right] = \sum_{i=0}^{n-1} \mu = n\mu$$

- **variance:**

$$\mathbb{V}\left(r_{t,t+n-1}^{log}\right) = \mathbb{V}\left(\sum_{i=0}^{n-1} r_{t+i}^{log}\right) = \sum_{i=0}^{n-1} \mathbb{V}\left(r_{t+i}^{log}\right) + \sum_{i\neq j}^{n-1} Cov\left(r_{t+i}^{log}, r_{t+j}^{log}\right)$$

$$= \sum_{i=0}^{n-1} \mathbb{V}\left(r_{t+i}^{log}\right) + 0 = n\sigma^2$$

- **standard deviation:**

$$\sigma_{t,t+n-1} = \sqrt{\mathbb{V}\left(r_{t,t+n-1}^{log}\right)} = \sqrt{n\sigma^2} = \sqrt{n}\sigma$$

Distribution of multi-period returns

- assumption: $r_t^{log} \sim \mathcal{N}(\mu, \sigma)$
- consequences:
  - random vector $\left(r_t^{log}, r_{t+k}^{log}\right)$ follows a **bivariate normal distribution** with zero correlation because of assumed independence

$$\left(r_t^{log}, r_{t+k}^{log}\right) \sim \mathcal{N}_2\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\right)$$

  - as a **sum** of components of a multi-dimensional **normally distributed** random vector, **multi-period returns** are **normally** distributed themselfes
- using formulas for multi-period moments we get

$$r_{t,t+n-1}^{log} \sim \mathcal{N}\left(n\mu, \sqrt{n}\sigma\right)$$

Multi-period VaR

- notation:
  - $\mu_n := \mathbb{E}\left[r^{log}_{t,t+n-1}\right] = n\mu$
  - $\sigma_n := \sigma_{t,t+n-1} = \sqrt{n}\sigma$
  - $VaR^{(n)}_\alpha := VaR_\alpha\left(r^{log}_{t,t+n-1}\right)$
- rewriting VaR_α for multi-period returns as function of one-period VaR_α :

$$\begin{aligned} VaR^{(n)}_\alpha &= -\mu_n + \sigma_n \Phi^{-1}(\alpha) \\ &= -n\mu + \sqrt{n}\sigma\Phi^{-1}(\alpha) \\ &= -n\mu + \sqrt{n}\mu - \sqrt{n}\mu + \sqrt{n}\sigma\Phi^{-1}(\alpha) \\ &= \left(\sqrt{n} - n\right)\mu + \sqrt{n}\left(-\mu + \sigma\Phi^{-1}(\alpha)\right) \\ &= \left(\sqrt{n} - n\right)\mu + \sqrt{n} VaR_\alpha\left(r^{log}_t\right) \end{aligned}$$

Multi-period ES

$$
\begin{aligned}
ES_\alpha^{(n)} &= -\mu_n + \sigma_n \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \\
&= -n\mu + \sqrt{n}\sigma \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \\
&= \left(\sqrt{n} - n\right)\mu + \sqrt{n}\left(-\mu + \sigma \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\right) \\
&= \left(\sqrt{n} - n\right)\mu + \sqrt{n} ES_\alpha
\end{aligned}
$$

- again, for $\mu = 0$ the **square-root-of-time scaling** applies:

$$
ES_\alpha^{(n)} = \sqrt{n} ES_\alpha
$$

Multi-period calculation example

$$\begin{aligned} VaR^{(3)}_{(1),0.99} &= \left(\sqrt{n} - n\right)\mu_1 + \sqrt{n} VaR^{(1)}_{(1),0.99} \\ &= \left(\sqrt{3} - 3\right) \cdot 0.039 + \sqrt{3} \cdot 2.535 \\ &= 4.342 \end{aligned}$$

$$\begin{aligned} ES^{(10)}_{(2),0.999} &= \left(\sqrt{n} - n\right)\mu_2 + \sqrt{n} ES^{(10)}_{(2),0.999} \\ &= \left(\sqrt{10} - 10\right) \cdot 0.024 + \sqrt{10} \cdot 5.684 \\ &= 17.8079 \end{aligned}$$

Multi-period value-at-risk

- control results for value-at-risk for 3 and 10 periods
  - $VaR^{(3)}_{(1),0.95} = 3.036, \quad VaR^{(3)}_{(1),0.99} = 4.342, \quad VaR^{(3)}_{(1),0.999} = 5.806$
  - $VaR^{(10)}_{(1),0.95} = 5.368, \quad VaR^{(10)}_{(1),0.99} = 7.752, \quad VaR^{(10)}_{(1),0.999} = 10.425$
  - $VaR^{(3)}_{(2),0.95} = 4.757, \quad VaR^{(3)}_{(2),0.99} = 6.758, \quad VaR^{(3)}_{(2),0.999} = 9.001$
  - $VaR^{(10)}_{(2),0.95} = 8.576, \quad VaR^{(10)}_{(2),0.99} = 12.229, \quad VaR^{(10)}_{(2),0.999} = 16.324$

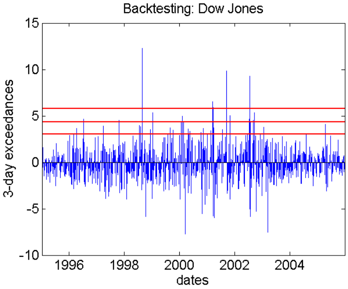
Multi-period expected shortfall

- control results for expected shortfall values for 3 and 10 periods
  - $ES^{(3)}_{(1),0.95} = 3.837, \quad ES^{(3)}_{(1),0.99} = 4.992, \quad ES^{(3)}_{(1),0.999} = 6.337$
  - $ES^{(10)}_{(1),0.95} = 6.830, \quad ES^{(10)}_{(1),0.99} = 8.938, \quad ES^{(10)}_{(1),0.999} = 11.394$
  - $ES^{(3)}_{(2),0.95} = 5.984, \quad ES^{(3)}_{(2),0.99} = 7.753, \quad ES^{(3)}_{(2),0.999} = 9.814$
  - $ES^{(10)}_{(2),0.95} = 10.816, \quad ES^{(10)}_{(2),0.99} = 14.045, \quad ES^{(10)}_{(2),0.999} = 17.808$

Multi-period backtesting: 3 days

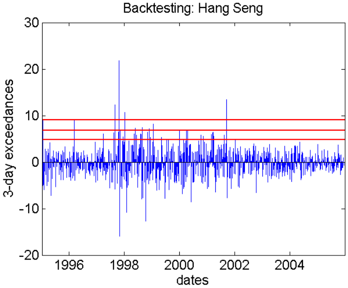
- in order to treat exceedances as independent realizations of binomial variable with probability of occurrence equal to confidence level, non-overlapping 3-day returns have to be considered

target	0.05	0.01	0.001
exceedances	0.044	0.017	0.007



Multi-period backtesting: 3 days

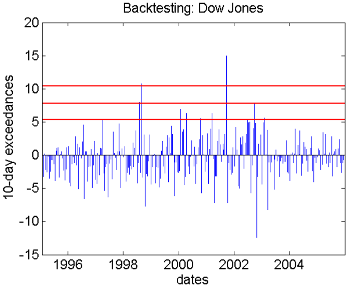
target	0.05	0.01	0.001
exceedances	0.043	0.016	0.008



Multi-period backtesting: 10 days

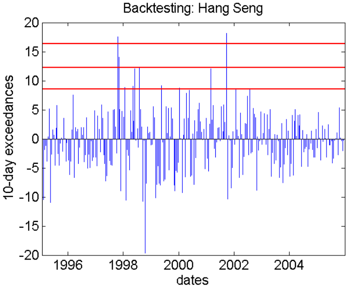
- sample size decreases with increasing time span of non-overlapping multi-period returns

target	0.05	0.01	0.001
exceedances	0.034	0.015	0.008



Multi-period backtesting: 10 days

target	0.05	0.01	0.001
exceedances	0.042	0.011	0.008



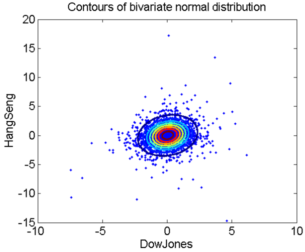
Out-of-sample performance

target	0.05	0.01	0.001
exceed. DJ 3 days	0.066	0.020	0.011
exceed. DJ 10 days	0.053	0.038	0.015
exceed. HS 3 days	0.043	0.027	0.009
exceed. HS 10 days	0.053	0.030	0.015

- again: notice the bad performance especially for higher confidence levels

Joint distribution

- contour plots of joint normal distribution together with empirical data of assets:



- note: some outliers seem to contradict the assumption of joint normally distributed returns

Calculating portfolio returns

- conversion between single asset logarithmic and discrete returns:

$$\log\left(1 + r^{discr}\right) = r^{log}$$

$$exp\left(r^{log}\right) - 1 = r^{discr}$$

- for discrete returns, the **two-asset portfolio return** is calculated by

$$r_P^{discr} = w \cdot r_A^{discr} + (1 - w) \cdot r_B^{discr}$$

- for logarithmic returns, we get

$$\begin{aligned} r_P^{log} &= \log(r_P^{discr} + 1) \\ &= \log\left(w \cdot r_A^{discr} + (1 - w) \cdot r_B^{discr} + 1\right) \\ &= \log\left(w \cdot \left(exp\left(r_A^{log}\right) - 1\right) + (1 - w)\left(exp\left(r_B^{log}\right) - 1\right) + 1\right) \\ &= \log\left(w \cdot exp\left(r_A^{log}\right) + (1 - w) \cdot exp\left(r_B^{log}\right)\right) \end{aligned}$$

Calculating portfolio returns

- while the **discrete** portfolio return can be calculated as **weighted sum**, the **logarithmic** portfolio return is a **non-linear function** of the individual asset returns
- however, in practice one often uses the **approximation**

$$r_P^{log} = w \cdot r_A^{log} + (1 - w) \cdot r_B^{log}$$

- this way, portfolio returns of normally distributed assets remain normally distributed

Approximation errors

- given asset weights $w = 0.5$ and $r_A^{log} = 0.08$ and $r_B^{log} = 0.04$, the correct logarithmic portfolio return is

$$r_P^{log} = \log\left(0.5 \cdot exp\left(0.08\right) + 0.5 \cdot exp\left(0.04\right)\right) = 0.0602$$

  in contrast to

$$r_P^{log} \approx 0.5 \cdot 0.08 + 0.5 \cdot 0.04 = 0.06$$

- however, for returns $r_A^{log} = 0.12$ and $r_B^{log} = 0$ the approximation is given by

$$r_P^{log} \approx 0.5 \cdot 0.12 + 0.5 \cdot 0 = 0.06,$$

  although the correct logarithmic return of the second portfolio results in

$$r_P^{log} = \log\left(0.5 \cdot exp\left(0.12\right) + 0.5 \cdot exp\left(0\right)\right) = 0.0618$$

- using the approximation, **qualitative statements** are **distorted**

Moments under approximation

- portfolio expectation:

$$\mathbb{E}\left[r_P^{log}\right] \approx \mathbb{E}\left[w \cdot r_A^{log} + (1-w) \cdot r_B^{log}\right]$$
$$= w\mathbb{E}\left[r_A^{log}\right] + (1-w)\,\mathbb{E}\left[r_B^{log}\right]$$
$$= w\mu_A + (1-w)\,\mu_B$$

- portfolio variance:

$$\mathbb{V}\left(r_P^{log}\right) \approx w^2\mathbb{V}\left(r_A^{log}\right) + (1-w)^2\,\mathbb{V}\left(r_B^{log}\right) + 2w\,(1-w)\;Cov\left(r_A^{log}, r_B^{log}\right)$$

$$\sigma_P \approx \sqrt{w^2\mathbb{V}\left(r_A^{log}\right) + (1-w)^2\,\mathbb{V}\left(r_B^{log}\right) + 2w\,(1-w)\;Cov\left(r_A^{log}, r_B^{log}\right)}$$

- under normal assumption:

$$r_P^{log} \sim \mathcal{N}\left(w\mu_A + (1-w)\,\mu_B, \sigma_P\right)$$

Influence of correlation

- since

$$\sigma_P = \sqrt{w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\,Cov(r_1, r_2)}$$
$$= \sqrt{w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\,\rho\sigma_1\sigma_2},$$

  and

$$VaR_{P,\alpha} = -\mu_P + \sigma_P\Phi^{-1}(\alpha),$$

  the portfolio standard deviation as well as the portfolio $VaR_{P,\alpha}$ are **increasing functions of ρ**

- explanation: high values of ρ lead to simultaneous increases or decreases of both assets, so that diversification effects are low

Approximation of portfolio VaR

- requirements:
  - logarithmic portfolio returns are approximated by weighted sum of individual logarithmic asset returns
  - assets are jointly normally distributed:
    - both individual assets are normally distributed
    - the dependence structure between both assets is of linear nature
- consequence: portfolio returns are normally distributed
- simple analytical formulas for *VaR* and *ES* under normally distributed returns are applicable

Calculations

- calculate portfolio expectation for weights $w = 0.5$:

$$\mu_P = 0.5 \cdot 0.039 + 0.5 \cdot 0.024 = 0.0315$$

- calculate portfolio standard deviation for $Cov = 0.309$:

$$\begin{aligned}\sigma_P &= \sqrt{0.5^2\sigma_1^2 + 0.5^2\sigma_2^2 + 2 \cdot 0.5^2 Cov\,(r_1, r_2)} \\ &= \sqrt{1.179}\end{aligned}$$

- calculate portfolio $VaR_{P,0.95}$ according to formula

Risk measure calculations

- note: out-of-sample exceedance frequencies are 14 times higher than expected
- indication for misspecified model

α	0.95	0.99	0.999
$VaR_{P,\alpha}$	1.755	2.495	3.324
$ES_{P,\alpha}$	2.208	2.863	3.625
backtest frequ.	0.045	0.018	0.006
Out-of-sample frequ.	0.062	0.028	0.014

Alternative way

- based on the simplifying assumptions made so far, we know how to calculate $VaR_{P,\alpha}$ via the following way:
  - approximate portfolio distribution by weighted sum of individual asset returns
  - derive $VaR_{P,\alpha}$ from portfolio distribution
- however, we also want to arrive at $VaR_{P,\alpha}$ via:
  - calculate $VaR_{1,\alpha}$ and $VaR_{2,\alpha}$ for individual assets 1 and 2
  - derive $VaR_{P,\alpha}$ as function of individual assets: $VaR_{P,\alpha} = f\left(VaR_{1,\alpha}, VaR_{2,\alpha}\right)$

Further simplifications

- in addition to the assumptions made so far, also assume individual expected asset returns to be zero:

$$\mu_1 = \mu_2 = 0$$

- expected portfolio return also vanishes:

$$\mu_P = w\mu_1 + (1 - w)\,\mu_2 = 0$$

- using

$$\sigma_P^2 = w^2\sigma_1^2 + (1 - w)^2\,\sigma_2^2 + 2\rho\sigma_1\sigma_2$$

we get:

$$\begin{aligned} VaR_{P,\alpha}^2 &= \sigma_P^2 \left(\Phi^{-1}(\alpha)\right)^2 \\ &= w_1^2\sigma_1^2 \left(\Phi^{-1}(\alpha)\right)^2 + (1 - w)^2\,\sigma_2^2 \left(\Phi^{-1}(\alpha)\right)^2 + 2\rho\sigma_1\sigma_2 \left(\Phi^{-1}(\alpha)\right)^2 \\ &= w_1^2 \cdot VaR_{1,\alpha}^2 + (1 - w)^2 \cdot VaR_{2,\alpha}^2 + 2\rho \cdot VaR_{1,\alpha} \cdot VaR_{2,\alpha} \end{aligned}$$

Consequences

- now we have

$$VaR_{P,\alpha} = \sqrt{w_1^2 \cdot VaR_{1,\alpha}^2 + (1-w)^2 \cdot VaR_{2,\alpha}^2 + 2\rho \cdot VaR_{1,\alpha} \cdot VaR_{2,\alpha}}$$
$$= f\left(VaR_{1,\alpha}, VaR_{2,\alpha}\right)$$

- purpose:
  - in general it can be very difficult to correctly model the portfolio distribution: derivation involves convolution applied to multi-dimensional distribution of all assets in the portfolio
  - in contrast to that, VaR_i for individual assets is rather easy to determine
  - deriving the overall portfolio VaR_P as simple function of individual VaR_i and correlation is very tempting
- note: as the assumptions used during the derivation of the formula are not fulfilled in reality, calculating VaR_P this way is just another approximation to the real value