Slides for Risk Management
Credit Risk

Groll

Seminar für Finanzökonometrie

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Definition

Definition

Credit risk predominantly comprises the risk of losses arising from an inability of a counterparty of a financial contract to fulfill promised payments. The case of a counterparty failing to meet its financial obligations is called **default**.

- in a broader context credit risk also is perceived as entailing
  - **credit spread** risk
  - **downgrade** risk

Default risk

- each counterparty is assumed to have an inherent **probability of default**, which can **not** be directly **observed**
- in order to assess the risk arising from this possible event of default, the following **quantitative characteristics** have to be estimated for each counterparty:
    - **probability of default (PD)**: the probability of a default event
    - **exposure at default (EAD)**: the amount that still has to be repayed by the borrower at the time of his default (could be random)
    - **loss given default (LGD)**: the fraction of the still outstanding obligations that the borrower is not able to repay, given that he defaults

Default risk

- the loss arising from a counterparty is given by

$$L = \underbrace{EAD \cdot LGD}_{\text{amount of money}} \cdot \underbrace{\mathbf{1}_D}_{\text{default indicator}},$$

  where D denotes the event of default with associated probability $\mathbb{P}(D) = PD$

- assuming **independence** between the amount of loss $EAD \cdot LGD$ and the occurrence of default $\mathbf{1}_D$, the **expected loss (EL)** can be calculated as

$$\begin{aligned}\mathbb{E}[L] &= \mathbb{E}[EAD \cdot LGD \cdot \mathbf{1}_D] \\ &= \mathbb{E}[EAD \cdot LGD] \cdot \mathbb{E}[\mathbf{1}_D] \\ &= \mathbb{E}[EAD \cdot LGD] \cdot PD\end{aligned}$$

Simplifications

- argumentation against independence: times of **financial turmoil** both increase likelihood of default events and amounts of losses, since remaining firm assets usually achieve less liquidation value because of low market prices caused by low demand due to bad market conditions: LGD and $\mathbf{1}_D$ should **not** be **independent**
- **assumption**: EAD is not a random variable
- this does not hold for financial contracts with **varying** underlying exposure like in the case of market-driven instruments such as **swaps and forwards**

Example

A bank has granted a loan with nominal value 100 € to an industrial company. The company in turn has used the borrowed money to bring up enough capital for an investment in a new product line. However, the success of the new product line and hence the size of the future profits are uncertain. Hence, the ability of the company to repay the loan depends on the success of the investment project, which has the following **distribution of profits**:

profit	10	40	80	100	150
probability	1%	2%	3%	4%	90%

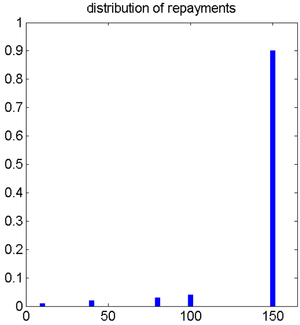
Given $EAD = 100$, assess the risk arising to the bank by calculation of

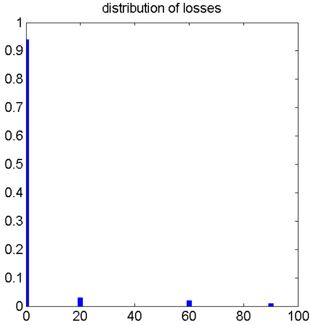
$$1. PD \quad 2. EL \quad 3. LGD \quad 4. VaR_{0.95}.$$

Example: loss distribution

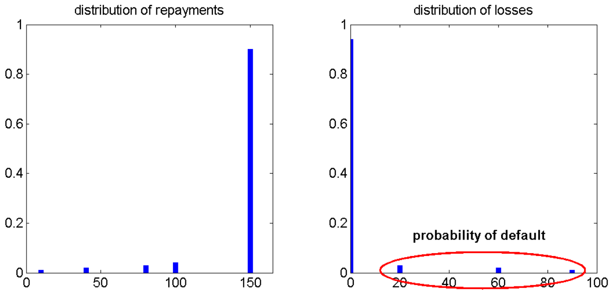
- given repayment C for the investment, the associated loss to the bank is given by

$$L = \max\{EAD - C, 0\}$$



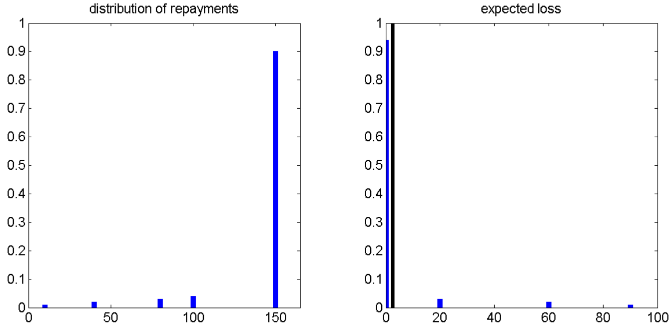


Example: PD



$$PD = 3\% + 2\% + 1\% = 6\%$$

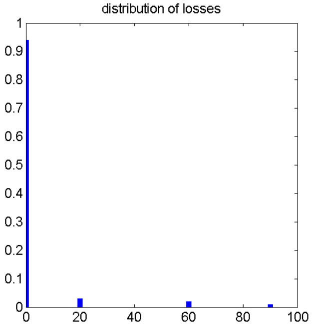
Example: EL



$$EL = 94\% \cdot 0 + 3\% \cdot 20 + 2\% \cdot 60 + 1\% \cdot 90 = 2.7$$

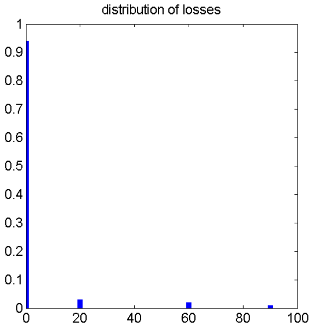
Example: LGD

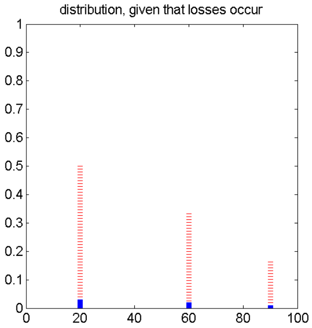
- since default is given, outcomes without loss are excluded



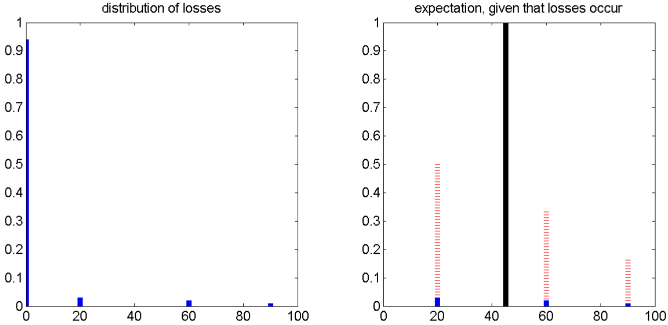
Example: LGD

- probabilities of events have to be scaled up in order to sum up to probability 1





Example: LGD


distribution of losses
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
20
40
60
80
100
expectation, given that losses occur
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
20
40
60
80
100

Example: LGD

$$
\begin{aligned}
LGD &= \mathbb{E}\left[L \mid L > 0\right] \\
&= \frac{\mathbb{P}(L = 20)}{\mathbb{P}(L > 0)} \cdot 20 + \frac{\mathbb{P}(L = 60)}{\mathbb{P}(L > 0)} \cdot 60 + \frac{\mathbb{P}(L = 90)}{\mathbb{P}(L > 0)} \cdot 90 \\
&= \frac{0.03}{0.06} \cdot 20 + \frac{0.02}{0.06} \cdot 60 + \frac{0.01}{0.06} \cdot 90 \\
&= 45
\end{aligned}
$$

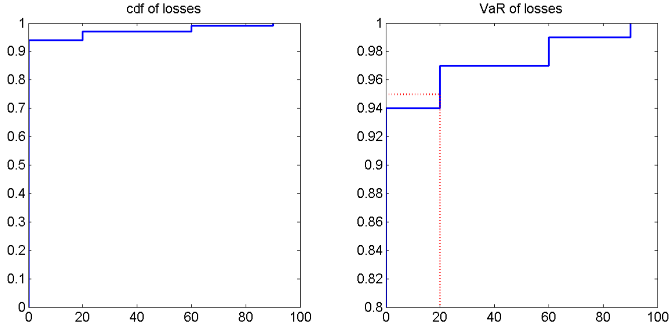
Example: VaR

- cumulative distribution of losses

loss L	0	20	60	90
probability $\mathbb{P}(L \leq l)$	94%	97%	99%	100%

Example: VaR

- getting $VaR_{0.95}$



Present value

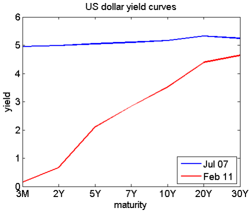
- given **certain** payments $c_1, \dots, c_n$ at future points in time $t+1, \dots t+n$, the **present value** V_t of the future payments is calculated by discounting:

$$
\begin{aligned}
V_t &= \frac{c_1}{(1+r_{t+1})} + \frac{c_2}{(1+r_{t+1})(1+r_{t+2})} + \cdots + \frac{c_n}{(1+r_{t+1})\cdots(1+r_{t+n})} \\
&= \frac{c_1}{(1+R_{t+1})} + \frac{c_2}{(1+R_{t+2})^2} + \cdots + \frac{c_n}{(1+R_{t+n})^n},
\end{aligned}
$$

where r_{t+i} denotes the (discrete) interest rate between $(t+i-1)$ and $(t+i)$, and R_{t+n} denotes the (discrete) annualized interest rate for periods $t+1$ to $t+n$

Yield curve

- based on **observable prices** for investment opportunities that are generally considered **riskless** (government bonds of industrial countries), one can extract prevailing market interest rates in a **yield curve**



- given the prevailing interest rates, the present value of any new stream of cash flows can be calculated

Valuation under uncertainty

- leaving the world of riskless and guaranteed future payments: cash flow c_i only occurs in case of state D_i, with probability $p_i := \mathbb{P}(D_i \text{ occurs})$
- what about the present value of the stream of **uncertain future cash flows**?
- regarding the problem as a sum of uncertain present values,

$$\sum_{i=1}^{n} \mathbf{1}_{D_i} \cdot V_t^{(i)} = \sum_{i=1}^{n} \mathbf{1}_{D_i} \cdot \frac{c_i}{(1 + R_{t+i})^i},$$

  the problem can be reduced to a **lottery in the present**
- the **expected payoff** of this lottery can be calculated straightforward:

$$\mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}_{D_i} \cdot V_t^{(i)}\right] = \sum_{i=1}^{n} \mathbb{P}(D_i \text{ occurs}) \cdot V_t^{(i)} = \sum_{i=1}^{n} p_i \cdot V_t^{(i)}$$

- however: the **present value** that people assign to the **lottery** over a number of uncertain values does **not equal** its **expectation**

Risk aversion

- if one asks people to participate in a game where they win 1€ in case of heads at a coin toss, but lose 1€ in case of tail, it would not be difficult to find people that are willing to play
- however, **increasing the bet** to 10000€ instead of 1€, people in general would **not** be **willing to participate** in the game, unless they get an adjustment to their favor
- the adjustment demanded as a compensation for the risk involved in the game is called **risk premium**, depending both on **the odds** of the game, as well as on the **amount of money** involved in case of adverse outcomes

Valuation under uncertainty

- transfering the perception of risk aversion to the valuation of uncertain payoffs requires that each term has to be devaluated additionally to discounting

$$V_t = \sum_{i=1}^{n} \frac{c_i}{f(c_i, p_i)(1 + R_{t+i})^i},$$

  which usually is written as an additional **spread** s_i on the riskless interest rate:

$$V_t = \sum_{i=1}^{n} \frac{c_i}{(1 + R_{t+i} + s_i)^i}$$

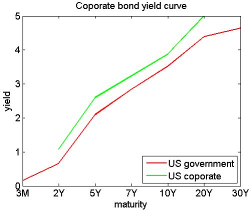
- according to **arbitrage theory** compensation for risk has to take place in a consistent manner across the market (risk neutral martingal measure)

Additional risk components

- besides the risk of a default of the borrower, holding a bond (or loan) also entails the **risk of a depreciation**, which comes into play if it shall be **sold prior to maturity** and default
- there are three components in the valuation of a bond that can cause depreciations:
  - **interest rate risk**: depreciations caused by increases of the discount interest rate
  - **downgrade risk**: the probability of default of the borrower increases leading to a higher demanded risk compensation when reselling the bond
  - **spread risk**: changes in the prevailing risk aversion of investors that lead to higher demanded risk compensations besides unchanged probability of default

Zero rate curves

- in reality, neither exact default probabilities nor exact risk premiums can be individually observed in the market
- however, given the bonds of several firms with approximately **equal default probabilities**, an **associated yield curve** can be inferred on the basis of the observable bond prices
- hence, even without knowledge about exact probabilities of default or associated risk premiums individually, the **value of** any **new bond** with comparable risk characteristics **can be calculated**

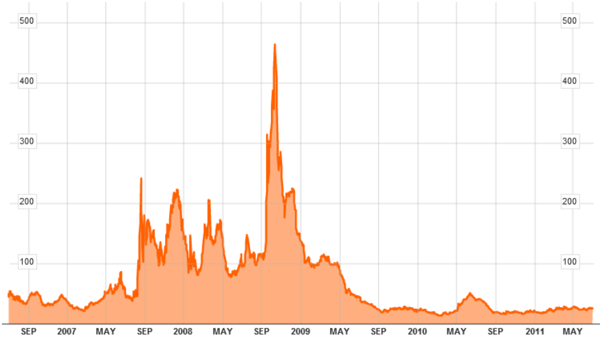


TED spread

- **TED spread** (Treasury Bill Eurodollar Difference): difference between interest rates on three-months interbank loans and three-months U.S. government debt ("T-bills")
- **LIBOR** (London Interbank Offered Rate): interest rates at which banks borrow unsecured funds of each other in the London money market
- while three-months **U.S. government debt** is largely considered riskless, overnight lending in the **interbank** money market involves **default risk**
- TED spread measures additional **risk premium demanded** in interbank money markets
- **example**: T-bill rates of 3.40% and LIBOR rates of 3.70% lead to TED spread of 30 (denoted in bps)

TED spread

- two possible explanations for **increasing** TED spread
  - loss in confidence of the credit worthiness of banks: **default probabilities** have **increased**
  - compensation demanded for any given portion of risk has increased: **risk aversion** in the market has **increased**



Example: revaluation

A bank holds a corporate bond with principal 100€, annual coupon payments of 6€ and maturity 3 years in its portfolio. According to the internal rating system of the bank, the issuer of the bond can be classified as "reliable" borrower. However, since the bank is planning to resell the bond within the next months, the riskmanagement division shall assess the loss associated with a decreasing credit quality of the borrower and an associated downgrade to the rating category "unreliable". Yields for both internal rating categories are given by

maturity in years	1	2	3
"reliable"	2%	2.8%	3.2%
"unreliable"	3.4%	4.8%	5.4%

Example: revaluation

- future cash flows associated with the bond in case of no default:

period	$t=1$	$t=2$	$t=3$
cash flow	6	6	100+6

- calculating the current value of the bond:

$$P_t^{reliable} = \frac{6}{(1.02)} + \frac{6}{(1.028)^2} + \frac{106}{(1.032)^3} = 108.002$$

- calculating value in case of downgrade:

$$P_t^{unreliable} = \frac{6}{(1.034)} + \frac{6}{(1.048)^2} + \frac{106}{(1.054)^3} = 101.794$$

Main risk components

- **default risk**:
  - counterparty actually failing to meet its financial obligations
  - realizing **only in case of default**, when debt is still not resold to third party
  - determining factor: **default probability**
- **downgrade risk**:
  - losses induced by decreasing credit quality
  - realizing **even without actual default** in case of debt reselling
  - determining factor: **credit quality changes** due to fluctuations of default probabilities
- problem: default probabilities are **not observable**, and fluctuations in default probabilities hence all the less

Estimation methodologies

- **estimation methodologies** differ, depending on the publicly available market information about the borrowing firm:
    - without listed stocks, traded debt (bonds) or available credit rating: analysis based on **fundamental values** and quantitative business ratios derived from financial statement (e.g. Altmann's Z-score)
    - available **credit rating**: derive default probability from historic default rates of equally rated firms
    - **traded bonds**: derived from credit spreads
    - **listed stocks**: derived from asset value model using observed equity prices (Merton model)

Rating definitions Moody's

Long-Term Corporate Obligation Ratings:

- **Aaa** Obligations rated Aaa are judged to be of the **highest quality**, with minimal credit risk.
- **Aa** Obligations rated Aa are judged to be of high quality and are subject to very low credit risk.
- **A** Obligations rated A are considered upper-medium grade and are subject to low credit risk.
- **Baa** Obligations rated Baa are subject to moderate credit risk. They are considered medium-grade and as such **may possess certain speculative characteristics**.

Rating definitions Moody's

- **Ba** Obligations rated Ba are judged to have speculative elements and are subject to **substandtial credit risk**.
- **B** Obligations rated B are considered **speculative** and are subject to **high credit risk**.
- **Caa** Obligations rated Caa are judged to be of poor standing and are subject to very high credit risk.
- **Ca** Obligations rated Ca are highly speculative and are **likely in, or very near, default**, with some prospect of recovery of principal and interest.
- **C** Obligations rated C are the lowest rated class of bonds and are typically in default, with **little prospect for recovery of principal or interest**.

Distribution over classes

Distribution of European Bond Issuers by Whole Letter Rating 2009

Rating Category	percentage
Aaa	4.2
Aa	20.8
A	31.8
Baa	22.3
Ba	7.4
B	9.5
Caa-C	4.0
Investment-Grade	79.2
Speculative-Grade	20.8

Transition matrix

One-Year-Average Ratings Transition for Europe – 1985- 2009

	year end	Aaa	Aa	A	Baa	Ba	B	Caa-C	Defaults
initial rating	Aaa	91.1	8.37	0.39	0.03	0.09	0.01	0.01	0
	Aa	0.93	90.20	8.32	0.48	0.04	0	0.01	0.02
	A	0.03	4.16	89.80	5.55	0.26	0.03	0.03	0.14
	Baa	0	0.42	7.24	86.86	4.09	0.92	0.34	0.14
	Ba	0	0	0.75	7.39	78.81	10.75	1.18	1.13
	B	0	0	0.35	0.45	7.32	79.40	9.15	3.34
	Caa-C	0	0.27	0.06	0	0.74	10.64	70.61	17.69

Recovery Rates

- seniority of the bond determines its recovery rate in case of default
- revaluation: provide a credit spread corresponding to each category as well

Europe	1985-2009
Sr. Secured Loans	55.5
Sr. Unsecured Loans	43.0
Sr. Secured Bonds	38.7
Sr. Unsecured Bonds	24.5
Sr. Subordinated Bonds	34.3
Subordinated Bonds	25.4
Jr. Subordinated Bonds	n.a.

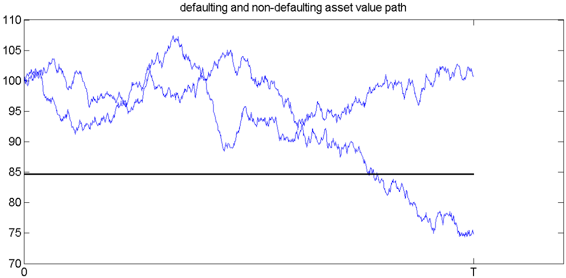
- standard deviation is missing

Introduction

- idea: use information incorporated in **equity prices** in order **to estimate default probabilities**
- first appearance in "On the pricing of corporate debt: The risk structure of interest rates" (Merton, 1974)
- asset value models: **default** occurs when value of firm's **assets** is **below** the nominal **debt** value
- two of most widely used credit risk models are based on asset value model interpretation:
  - KMV-Model: estimation of default probabilities from adjusted asset value model
  - CreditMetrics: estimation of default correlations based on asset value model

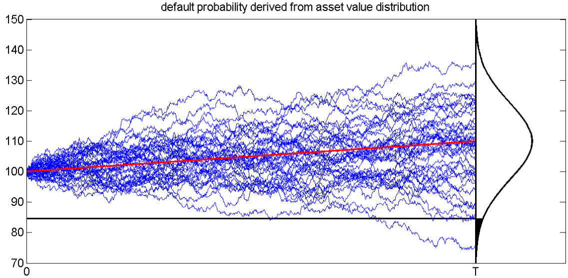
Asset-value - default connection

- with given liabilities the occurrence of default depends on the evolution of the firm's asset prices



Default probabilities

- with known **distribution of asset prices** at the end of the time horizon the **default probability** equals the fraction of asset price paths with values below the debt level



Problem

- although assets are listed in financial statements, market values for assets are usually not completely existing
- example: **market prices** of physical capital (machines, real estate, property) are **not existing**, simply because these assets are kept by the firm and are not traded
- based on the sparsity of information about asset values, estimating asset value dynamics seems impossible
- even exact numbers for **debt** may **not** be **known**: hidden financial obligations due to commitments, subsidiaries or contingent debt levels (swaps)
- solution: try to overcome asset value problems through incorporation of market data for equity

Equity as option

- assets A_t financed by equity E_t and debt D_t

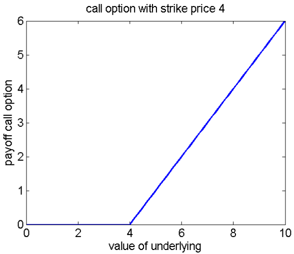
$$A_t = E_t + D_t$$

- share holders as the owners of the firm have the right to liquidate the firm at any time: paying off the debt and taking over the remaining assets
- two scenarios:
  - $A_T < D_t$: total value of assets are below financial obligations - **no assets left** for equity holders
  - $A_T \geq D_t$: after repaying debt equity holders are left with **net profit** of $A_T - D_T$
- payoff to equity holders is given by

$$\max\left(A_T - D_T, 0\right) = \left(A_T - D_T\right)^+$$

  equal to **call option** on asset values A_T with strike price D_T
- observable equity prices are the prices that investors are willing to pay for this payoff in T

Payoff call option



- goal: assume functional form for dynamics of asset values and **adjust** parameters of **dynamics to equity prices** observable in the market

Brownian motion

Definition

A real-valued stochastic process $(B_t)_{0 \leq t \leq \infty}$ is called **Brownian motion**, if

1. $B_0 = 0$
2. the function $t \to B_t(\omega)$ is continuous $\mathbb{P}$-a.s.
3. the increments $B_t - B_s$ are independent with distribution

$$B_t - B_s \sim \mathcal{N}(0, t - s)$$

for any $0 \leq s < t$.

Brownian motion

- **changing volatility** of Brownian motion: because of

$$\mathbb{V}(\sigma B_t) = \sigma^2 \mathbb{V}(B_t) = \sigma^2 t$$

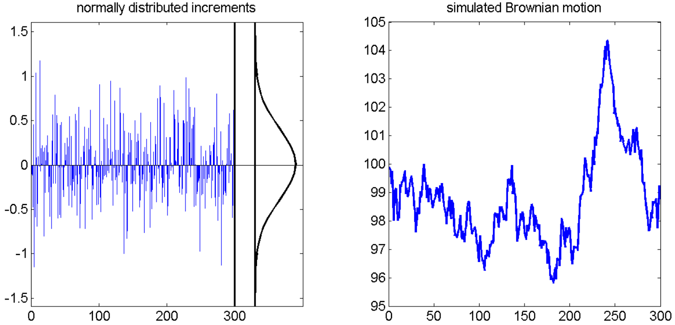
  the increments of the rescaled Brownian motion $(\sigma B_t)_{0 \leq t < \infty}$ are distributed according to

$$\sigma (B_t - B_s) \sim \mathcal{N}(0, \sigma^2 (t - s))$$

- example: for $\sigma = 0.4$ the rescaled Brownian motion has variance $\sigma^2 = 0.4^2 = 0.16$ per time period

Brownian motion

- discrete approximation of Brownian motion



Brownian motion

- simulated paths with **finer resolution** can be obtained by using n normally distributed random variables with variance $\frac{1}{n}$: for $X_i \sim \mathcal{N}(0,1)$ the scaled random variable $\sigma X_i = \frac{1}{\sqrt{n}} X_i$ has variance $\mathbb{V}\left(\frac{1}{\sqrt{n}} X_i\right) = \frac{1}{n}\mathbb{V}(X_i) = \frac{1}{n}$, so that

$$
\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n} \sigma X_i\right) &= \sigma^2 \mathbb{V}\left(\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbb{V}(X_i) \\
&= \frac{1}{n} \cdot n \cdot 1 \\
&= 1
\end{aligned}
$$

Deterministic dynamics

- consider deterministic function as degenerated special case of stochastic processes

$$A_t = A_0 \cdot exp\left(rt\right)$$

- value changes of process per time evolving

$$\frac{dA_t}{dt} = \left(A_t\right)' = \frac{d\left(A_0 \cdot exp\left(rt\right)\right)}{dt} = A_0 \cdot exp\left(rt\right) \cdot r = A_t r$$

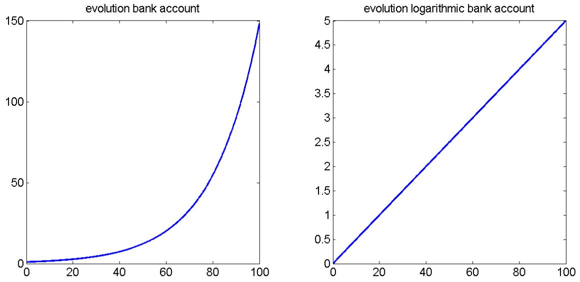
$$\Leftrightarrow \quad dA_t = rA_t dt$$

- intuitive interpretation: for any given value of the process, how does this process value change, given that we look at process for time dt
- example: given world population 7 billion, how does population number change in one year

Exponential growth

- dynamics of A_t on logarithmic scale

$$\frac{d\left(\log A_t\right)}{dt} = \left(\log A_t\right)' = \frac{1}{A_t} \cdot A_t' = \frac{1}{A_t} \cdot A_t \cdot r = r$$



- this dynamics is often used to model the evoluation of money on a bank account with constant rate of interest: given current account surplus of 1000€, how does amount of money change within next year?

Stochastic dynamics

- goal: process with **random changes**, with size of changes depending on current value

$$dA_t = A_t \sigma dB_t$$

- justification: absolute size of stock price changes depends on current stock price
- solution of stochastic differential equation:

$$A_t = A_0 \cdot exp\left(-\frac{1}{2}\sigma^2 t + \sigma B_t\right)$$

- proof: because of erratic behaviour of Brownian motion, approximation to changes of process must incorporate derivatives of higher order

Stochastic dynamics

- application of Ito's formula

$$dF(I_t) = F'(I_t)\, dI_t + \frac{1}{2}F''(I_t)\, d\langle I\rangle_t$$

to

$$A_t = A_0 \cdot exp\left(-\frac{1}{2}\sigma^2 t + \sigma B_t\right) = A_0 exp(I_t) = F(I_t)$$

leads to

$$\begin{aligned} dA_t &= exp\left(-\frac{1}{2}\sigma^2 t + \sigma B_t\right)\left(-\frac{1}{2}\sigma^2 dt + \sigma dB_t\right) + \frac{1}{2} exp\left(-\frac{1}{2}\sigma^2 t + \sigma B_t\right)\sigma^2 d\langle B\rangle_t \\ &= A_t \cdot \left(-\frac{1}{2}\sigma^2 dt + \sigma dB_t\right) + \frac{1}{2}\cdot A_t \sigma^2 dt \\ &= A_t \cdot \sigma dB_t \end{aligned}$$

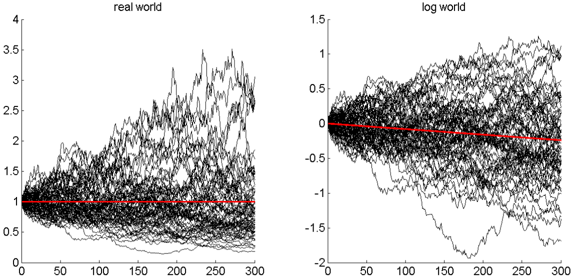
Stochastic dynamics

- on a logarithmic scale, with $(\log x)' = \frac{1}{x}$:

$$
\begin{aligned}
dS_t = d\left(\log A_t\right) &= \frac{1}{A_t} \cdot dA_t - \frac{1}{2} \cdot \frac{1}{A_t^2} d\left\langle A \right\rangle_t \\
&= \frac{1}{A_t} \cdot A_t \sigma dB_t - \frac{1}{2} \cdot \frac{1}{A_t^2} A_t^2 \sigma^2 dt \\
&= \sigma dB_t - \frac{1}{2} \sigma^2 dt
\end{aligned}
$$

Stochastic dynamics

- because of **convexity** of *exp*, a symmetric erratic behaviour on the logarithmic scale would lead to positive drift for real world asset prices
- hence, dynamics of logarithmic world show negative drift when expectation of real world is zero



Geometric Brownian motion

- incorporate drift term in real world dynamics

$$dA_t = A_t \mu dt + A_t \sigma dB_t$$

- solution:

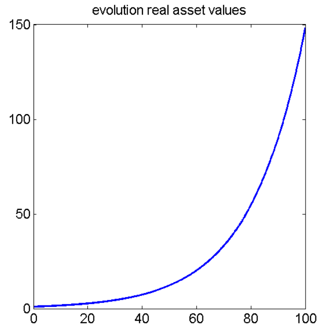
$$A_t = A_0 \cdot exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right)$$

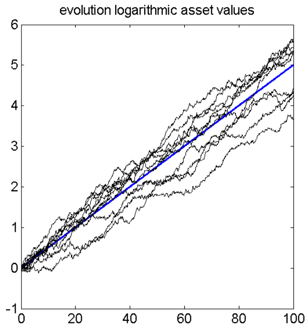
- dynamics on logarithmic scale

$$\begin{aligned} dS_t = d\left(\log A_t\right) &= \frac{1}{A_t} \cdot dA_t - \frac{1}{2} \cdot \frac{1}{A_t^2} d\left\langle A\right\rangle_t \\ &= \mu dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \end{aligned}$$

Geometric Brownian motion

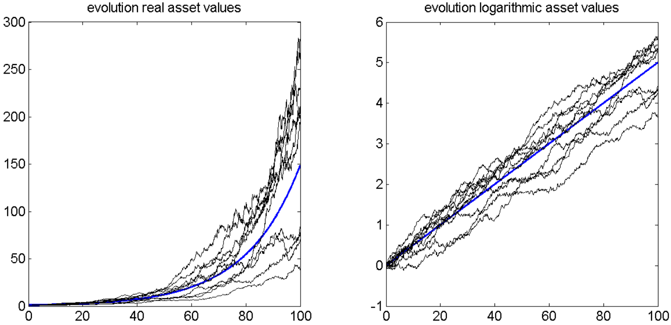
- goal: asset value expectation shall follow exponential growth, with random fluctuation attached
- first guess: introduce random fluctuations by attachment of Brownian motion in log-world





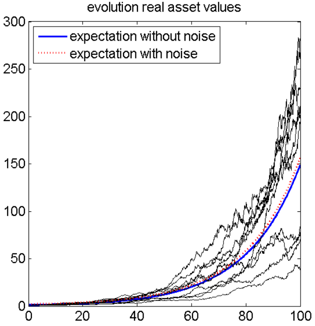
Geometric Brownian motion

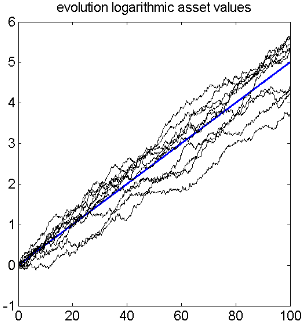
- convexity increases upward deviations



Geometric Brownian motion

- expectation in real world is greater than expectation in log-world
- hence: to get μ in real world, adjustment $-\frac{1}{2}\sigma^2$ is needed in log-world





Option pricing

- key assumptions:
  - asset value process follows a geometric Brownian motion
  - arbitrage-free world
- price of $C_T = (A_T - D_T)^+$ is less than $\mathbb{E}[C_T]$, since risk averse investors want compensation for risk
- according to arbitrage theory, the arbitrage-free price C_0 of C_T is given by the expectation of the discounted payoff C_T under a **risk-neutral equivalent measure** $\mathbb{Q}$:

$$C_0 = \mathbb{E}_{\mathbb{Q}}\left[\frac{C_T}{e^{rT}}\right]$$

- hence, the equity value E_t of a firm is related to the asset value process A_t and the debt value D_t by

$$E_0 = \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{A_T - D_T}{e^{rT}}\right)^+\right]$$

Dynamics under $\mathbb{Q}$

- changing to the risk-neutral measure $\mathbb{Q}$ nullifies excess returns above the risk-free interest rate:

$$dA_t = A_t r dt + A_t \sigma dB_t$$

  with solution

$$A_t = A_0 \cdot exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)$$

- additionally discounting leads to dynamics with expected return equal to zero:

$$d\left(\frac{A_t}{e^{rt}}\right) = \left(\frac{A_t}{e^{rt}}\right)\sigma dB_t$$

Black-Scholes formula

- denoting the discounted asset value process (A_t/e^{rt}) with $\hat{A}_t$, the solution to the dynamics under the risk-neutral measure is

$$\hat{A}_t = \hat{A}_0 \cdot exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$$

- denoting the discounted value of the debt by $\hat{D}_T$, the value of the call option of the assets $\hat{A}_t$ with strike price $\hat{D}_T$ is given by

$$\begin{aligned} C_0 &= \mathbb{E}\left[\left(\hat{A}_T - \hat{D}_T\right)^+\right] \\ &= \int_\Omega \left(\hat{A}_T - \hat{D}_T\right) \mathbf{1}_{\{\hat{A}_T > \hat{D}_T\}} d\omega \\ &= \int_{-\infty}^{\infty} \left(x - \hat{D}_T\right) \mathbf{1}_{\{x > \hat{D}_T\}} dF_{\hat{A}_T}(x) \end{aligned}$$

- hence, calculation of the expectation requires the cumulative distribution function and the probability density function of the random variable $\hat{A}_T$

Cumulative distribution function

- using knowledge of logarithmic asset value process

$$
\begin{aligned}
F_{\hat{A}_T}(x) &= \mathbb{P}\left(\hat{A}_T \leq x\right) \\
&= \mathbb{P}\left(\log \hat{A}_T \leq \log x\right)
\end{aligned}
$$

with

$$
\log \hat{A}_T = \log A_0 - \frac{1}{2}\sigma^2 T + \sigma B_T
$$

and

$$
\log \hat{A}_T \sim \mathcal{N}\left(\log A_0 - \frac{1}{2}\sigma^2 T, \sigma^2 T\right)
$$

$$
\Leftrightarrow \frac{\log \hat{A}_T - \log A_0 + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \sim \mathcal{N}(0,1)
$$

Cumulative distribution function

- plugging in:

$$
\begin{aligned}
F_{\hat{A}_T}(x) &= \mathbb{P}\left(\hat{A}_T \leq x\right) \\
&= \mathbb{P}\left(\log \hat{A}_T \leq \log x\right) \\
&= \mathbb{P}\left(\frac{\log \hat{A}_T - \log A_0 + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \leq \frac{\log x - \log A_0 + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\log x - \log A_0 + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\log\left(\frac{x}{A_0}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\
&:= \Phi\left(-h(x)\right), \text{ with } h(x) = \frac{-\log\left(\frac{x}{A_0}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}
\end{aligned}
$$

Probability density function

- density function is derivative of cdf:

$$\begin{aligned}
f_{\hat{A}_T}(x) &= F'_{\hat{A}_T}(x) \\
&= \left(\Phi\left(-h(x)\right)\right)' \\
&= \Phi'\left(-h(x)\right) \cdot \left(-h(x)\right)' \\
&= \phi\left(-h(x)\right) \cdot \left(\frac{1}{\sigma\sqrt{T}} \cdot \frac{1}{\frac{x}{A_0}} \cdot \frac{1}{A_0}\right) \\
&= \phi\left(-h(x)\right) \cdot \left(\frac{1}{x\sigma\sqrt{T}}\right),
\end{aligned}$$

because of

$$\left(-h(x)\right)' = \left(\frac{\log\left(\frac{x}{A_0}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)' = \left(\frac{\log\left(\frac{x}{A_0}\right)}{\sigma\sqrt{T}}\right)' + 0$$

- using density function we get

$$\begin{aligned}
\int_{\{\hat{A}_T \geq \hat{D}_T\}} \hat{A}_T dF_{\hat{A}_T} &= \int_{\hat{D}_T}^{\infty} xf(x)\,dx \\
&= \int_{\hat{D}_T}^{\infty} x \cdot \phi(-h(x)) \cdot \left(\frac{1}{x\sigma\sqrt{T}}\right) dx \\
&= \int_{\hat{D}_T}^{\infty} \frac{\phi(-h(x))}{\sigma\sqrt{T}} dx \\
&= \int_{\hat{D}_T}^{\infty} \frac{\phi(h(x))}{\sigma\sqrt{T}} dx \\
&\overset{(\star)}{=} A_0 \int_{\hat{D}_T}^{\infty} \frac{\phi(g(x))}{x\sigma\sqrt{T}} dx \\
&\overset{(\star\star)}{=} A_0 \left[-\Phi(g(x))\right]_{\hat{D}_T}^{\infty} \\
&= A_0 \Phi\left(g\left(\hat{D}_T\right)\right),
\end{aligned}$$

with

$$g(x) = h(x) + \sigma\sqrt{T}$$

- $(\star)$ holds because of

$$
\begin{aligned}
\phi(g(x)) &= \phi\left(h(x)+\sigma\sqrt{T}\right) \\
&= \frac{1}{\sqrt{2\pi}}\cdot exp\left(-\frac{\left(h(x)+\sigma\sqrt{T}\right)^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\cdot exp\left(-\frac{h(x)^2}{2}-\frac{2h(x)\sigma\sqrt{T}+\sigma^2 T}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\cdot exp\left(-\frac{h(x)^2}{2}\right)\cdot exp\left(-h(x)\sigma\sqrt{T}-\frac{\sigma^2 T}{2}\right) \\
&= \phi(h(x))\cdot exp\left(\frac{\log\left(\frac{x}{A_0}\right)+\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\cdot\sigma\sqrt{T}-\frac{\sigma^2 T}{2}\right) \\
&= \phi(h(x))\cdot exp\left(\log\left(\frac{x}{A_0}\right)\right) \\
&= \phi(h(x))\cdot\frac{x}{A_0},
\end{aligned}
$$

so that

$$\phi(h(x)) = \phi(g(x))\cdot\frac{A_0}{x}$$

- $(\star\star)$ holds because of

$$
\begin{aligned}
\left(-\Phi\left(g\left(x\right)\right)\right)' &= -\phi\left(g\left(x\right)\right)\cdot\left(g\left(x\right)\right)' \\
&= -\phi\left(g\left(x\right)\right)\cdot\left(h\left(x\right)+\sigma\sqrt{T}\right)' \\
&= -\phi\left(g\left(x\right)\right)\cdot\left(h\left(x\right)\right)' \\
&= \phi\left(g\left(x\right)\right)\cdot\left(\frac{1}{x\sigma\sqrt{T}}\right)
\end{aligned}
$$

- furthermore,

$$
\begin{aligned}
g\left(\infty\right) &= h\left(\infty\right)+\sigma\sqrt{T} \\
&= \frac{-\log\left(\frac{\infty}{A_0}\right)}{\sigma\sqrt{T}}-\frac{1}{2}\sigma\sqrt{T}+\sigma\sqrt{T} \\
&= -\frac{\infty}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T} \\
&= -\infty
\end{aligned}
$$

$$
\Rightarrow \quad \Phi\left(g\left(\infty\right)\right)=0
$$

Black-Scholes formula

$$
\begin{aligned}
\mathbb{E}[C] &= \int_{-\infty}^{\infty} \left(\hat{A}_T - \hat{D}_T\right)^{+} dF_{\hat{A}_T}(x) \\
&= \int_{-\infty}^{\infty} \left(\hat{A}_T - \hat{D}_T\right) \mathbf{1}_{\{\hat{A}_T > \hat{D}_T\}} dF_{\hat{A}_T}(x) \\
&= \int_{\{\hat{A}_T > \hat{D}_T\}} \hat{A}_T dF_{\hat{A}_T}(x) - \hat{D}_T \int_{\{\hat{A}_T > \hat{D}_T\}} dF_{\hat{A}_T}(x) \\
&= \int_{\hat{D}_T}^{\infty} \hat{A}_T dF_{\hat{A}_T}(x) - \hat{D}_T \mathbb{P}\left(\hat{A}_T > \hat{D}_T\right) \\
&= A_0 \Phi\left(g\left(\hat{D}_T\right)\right) - \hat{D}_T \cdot \Phi\left(h\left(\hat{D}_T\right)\right) \\
&= v\left(A_0, \hat{D}_T, T, \sigma\right)
\end{aligned}
$$

with

$$
h(x) = \frac{-\log\left(\frac{x}{A_0}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad g(x) = h(x) + \sigma\sqrt{T}
$$

Summary

- assumptions made by Black-Scholes world:
  - asset prices follow geometric Brownian motion
  - trading appears in continuous time
  - no transaction prices
  - borrowing and lending at risk-free rate is possible at arbitrarily high amounts

Summary

- given these assumptions, the probability of default is

$$
\begin{aligned}
\mathbb{P}(Y=1) &= \mathbb{P}(A_T \leq D_T) \\
&= \mathbb{P}(\log A_T \leq \log D_T) \\
&= \mathbb{P}\left(\log A_0 + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma B_T \leq \log D_T\right) \\
&= \mathbb{P}\left(\sigma B_T \leq \log D_T - \log A_0 - \left(r - \frac{1}{2}\sigma^2\right)T\right) \\
&= \mathbb{P}\left(\frac{B_T}{\sqrt{T}} \leq \frac{\log D_T - \log A_0 - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\log D_T - \log A_0 - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Summary

- according to option pricing theory, the observable equity prices E_0 is a function of the non-observable parameters A_0 and σ

$$E_0 = v\left(A_0, \hat{D}_T, T, \sigma\right)$$

- given we take a first estimate of σ, the only unknown parameter in the equation is the asset price
- locally inverting the option price formula hence gives an estimate of A_0
- repeating this procedures for a series of points in time with known equity prices gives an estimated time series of asset prices
- given the time series of asset prices, parameters σ and μ can be estimated, and can be used as a second guess input in the locally inverted option price formula
- repeating this procedure gives estimated values of σ and μ
- hence, PD can be estimated

Defaults as binomials

- given two **binomial** random variables X_1 and X_2 with

$$X_i \sim B(1;p), \quad i.e., \quad X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

- the **event** 1 is interpreted as **default**, with default probability

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_2 = 1) = p$$

- joint distribution for case of independence

		X_1		
		1	0	
X_2	0	$p(1-p)$	$(1-p)^2$	$(1-p)$
	1	p^2	$p(1-p)$	p
		p	$(1-p)$	

Default correlation

- notation

		X_1		
		1	0	
X_2	0	β	γ	$(1-p)$
	1	α	β	p
		p	$(1-p)$	

- calculate covariance

$$\begin{aligned} Cov\left(X_1, X_2\right) &= \mathbb{E}\left[X_1 \cdot X_2\right] - \mathbb{E}\left[X_1\right]\mathbb{E}\left[X_2\right] \\ &= \mathbb{P}\left(X_1 = 1, X_2 = 1\right) - p^2 \\ &= \alpha - p^2 \end{aligned}$$

- **default correlation**

$$\varrho_{X_1,X_2} = \frac{\alpha - p_1 p_2}{\sqrt{p_1\left(p_1-1\right)}\sqrt{p_2\left(p_2-1\right)}} \overset{p_1=p_2}{=} \frac{\alpha - p^2}{p\left(p-1\right)}$$

Example: independence

- given default probabilities $p_1 = p_2 = 0.1$, joint probabilities in case of independency are given by

		X_1		
		1	0	
X_2	0	**0.09**	**0.81**	0.9
	1	**0.01**	**0.09**	0.1
		0.1	0.9	

- covariance and correlation are given by

$$Cov\left(X_1, X_2\right) = \alpha - p^2 = 0.01 - 0.1^2 = 0$$

$$\varrho_{X_1, X_2} = \frac{Cov\left(X_1, X_2\right)}{p\left(1 - p\right)} = 0$$

Example: dependence

- the same individual default probabilities $p_1 = p_2 = 0.1$ also could lead to the following joint distribution

		X_1		
		1	0	
X_2	0	**0.02**	**0.88**	0.9
	1	**0.08**	**0.02**	0.1
		0.1	0.9	

- associated default correlation and covariance are

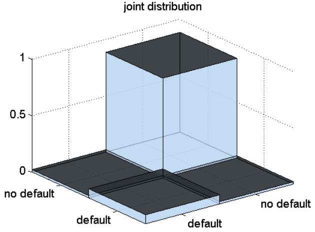
$$Cov\left(X_1, X_2\right) = 0.08 - 0.1^2 = 0.07$$

$$\varrho_{X_1,X_2} = \frac{Cov\left(X_1, X_2\right)}{p\left(1-p\right)} = \frac{0.07}{0.1 \cdot 0.9} = 0.778$$

- note: the probability of joint defaults is 8 times higher in the second case

Example: graphical representation

- expressing probabilities as pillars above the events {"default", "no default"}

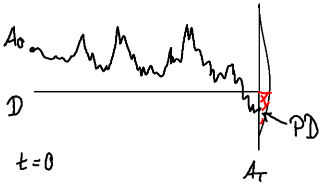


Example: interpretation as asset value model

Goal: find **joint default probability**

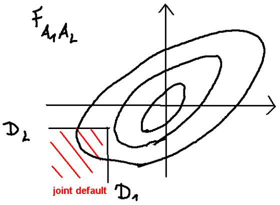
- thinking in terms of **asset value model**:
  - for $p = 0.1$, default happens in 10% **worst** possible **asset path** realizations
  - already known: for any given nominal debt value D_T the **probability of default depends** on the distribution of the **underlying asset value** by

$$p = \mathbb{P}(A_T < D_T)$$



Example: interpretation as asset value model

- the occurrence of **joint defaults** can be interpreted as **both** asset values lying **below** their respective debt level
- joint default in asset value model: **joint asset distribution** F required



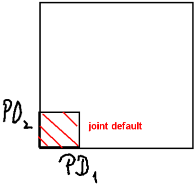
- joint default probability:

$$PD = F(D_1, D_2)$$

Example: interpretation as asset value model

- joint default probabilities can be equivalently described **in terms of quantiles**:

$$\begin{aligned} PD &= \mathbb{P}\left(A_1 < D_1, A_2 < D_2\right) \\ &= F\left(D_1, D_2\right) \\ &= C\left(F_1\left(D_1\right), F_2\left(D_2\right)\right) \\ &= C\left(PD_1, PD_2\right) \end{aligned}$$



Example: interpretation as asset value model

- hence, **given** that PD_i is already **known**:
  - the exact **marginal distribution** of assets does **not** provide **additional information**
  - joint default probabilities depend on asset pair copula only
  - marginal asset distributions need not be modelled
- joint **default distributions** can be **described by copula densities**

Example: associated copula density

- given joint distribution, density heights of an associated copula are calculated according to
  - both default

$$h_{1,1} = \frac{\mathbb{P}(X_1 = 1, X_2 = 1)}{\mathbb{P}(X_1 = 1) \cdot \mathbb{P}(X_2 = 1)} = \frac{0.08}{0.1 \cdot 0.1} = 8$$

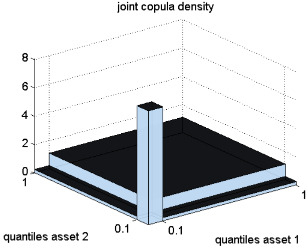
  - one default

$$h_{1,0} = h_{0,1} = \frac{\mathbb{P}(X_1 = 1, X_2 = 0)}{\mathbb{P}(X_1 = 1) \cdot \mathbb{P}(X_2 = 0)} = \frac{0.02}{0.1 \cdot 0.9} = 0.222$$

  - no default

$$h_{0,0} = \frac{\mathbb{P}(X_1 = 0, X_2 = 0)}{\mathbb{P}(X_1 = 0) \cdot \mathbb{P}(X_2 = 0)} = \frac{0.88}{0.9 \cdot 0.9} = 1.0864$$

Example: representation as copula density



- **given associated copula density**, probability of joint default can be calculated as probability mass in respective interval:

$$\mathbb{P}(X_1 = 1, X_2 = 1) = width_{X_1} \cdot width_{X_2} \cdot heights$$
$$= 0.1 \cdot 0.1 \cdot 8 = 0.08$$

Example: maximum dependence

- with given individual default probabilities $p_1 = p_2 = p$ the **maximum possible dependency** occurs when defaults always appear together:

		X_1		
		1	0	
X_2	0	**0**	**0.9**	0.9
	1	**0.1**	**0**	0.1
		0.1	0.9	

- associated default correlation and covariance are

$$Cov_{max}(X_1, X_2) = 0.1 - 0.1^2 = 0.09$$

$$\varrho_{max} = \frac{0.09}{0.09} = 1$$

Example: negative correlation

- with given individual default probabilities $p_1 = p_2 = p$ the **maximum diversification** effects occur, when defaults never appear together:

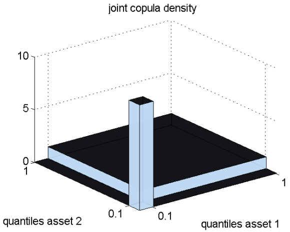
		X_1		
		1	0	
X_2	0	**0.1**	**0.8**	0.9
	1	**0**	**0.1**	0.1
		0.1	0.9	

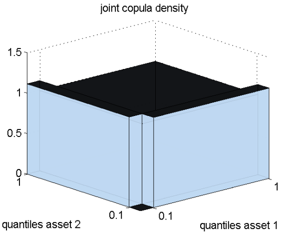
- associated default correlation and covariance are

$$Cov_{min}(X_1, X_2) = 0 - 0.1^2 = -0.01$$

$$\varrho_{min} = \frac{-0.01}{0.09} = -0.1111$$

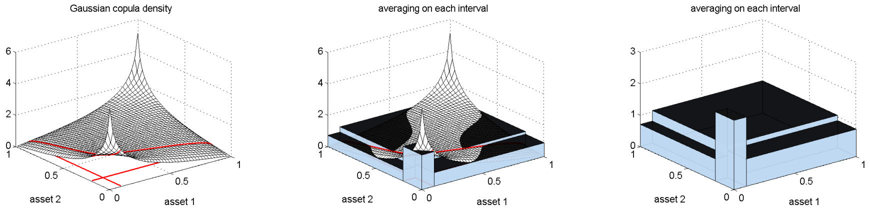
Graphical representation





Graphically derivation of joint default distribution

- for **given copula density**, partition unit cube according to individual default probabilities
- since exact asset path realizations are not necessary, individual intervals can be represented with uniform distribution of mean height



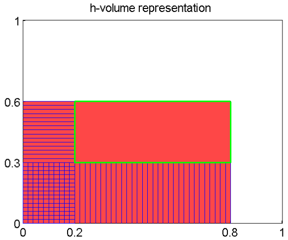
Analytical derivation

- given copula C, the probability mass in rectangle $[a_1, b_1] \times [a_2, b_2]$ is given by **copula h-volume**:

$$
\begin{aligned}
&\mathbb{P}\left(a_1 < A_1 \leq b_1, a_2 < A_2 \leq b_2\right) = \\
=&\mathbb{P}\left(A_1 \leq b_1, A_2 \leq b_2\right) - \mathbb{P}\left(A_1 \leq b_1, A_2 \leq a_2\right) \\
&- \mathbb{P}\left(A_1 \leq a_1, A_2 \leq b_2\right) + \mathbb{P}\left(A_1 \leq a_1, A_2 \leq a_2\right) \\
=&C\left(b_1, b_2\right) - C\left(b_1, a_2\right) - C\left(a_1, b_2\right) + C\left(a_1, a_2\right)
\end{aligned}
$$

h-volume derivation

- graphical representation of probability mass in $[a_1, b_1] \times [a_2, b_2]$



Example: Gaussian vs. Clayton copula

- let individual default probabilities be given by $p_1 = p_2 = 0.05$
- calculate **joint default** probability with **Gaussian copula** with parameter $\rho = 0.57$

$$\begin{aligned}\mathbb{P}(A_1 \leq 0.05, A_2 \leq 0.05) &= C^{Gau}(0.05, 0.05; \rho = 0.57) \\ &= 0.0145\end{aligned}$$

- calculate probability of **one default**

$$\begin{aligned}\mathbb{P}(A_1 \leq 0.05, A_2 > 0.05) &= \mathbb{P}(A_1 > 0.05, A_2 \leq 0.05) \\ &= p_1 - \alpha \\ &= 0.05 - 0.0145 \\ &= 0.0355\end{aligned}$$

Example: Gaussian vs. Clayton copula

- calculate probability of **no default**

$$\begin{aligned}\mathbb{P}(A_1 > 0.05, A_2 > 0.05) &= 1 - 2 \cdot \beta - \alpha \\ &= 1 - 2 \cdot 0.0355 - 0.0145 \\ &= 0.9145\end{aligned}$$

		X_1		
		1	0	
X_2	0	0.035	0.915	0.95
	1	0.015	0.035	0.05
		0.05	0.95	

Example: Gaussian vs. Clayton copula

- calculate **joint default** probability with **Clayton copula** with parameter $\alpha = 1.3$

$$\begin{aligned}\mathbb{P}\left(A_1 \leq 0.05, A_2 \leq 0.05\right) &= C^{Clay}\left(0.05, 0.05; \alpha = 1.3\right) \\ &= \left(u_1^{-\alpha} + u_2^{-\alpha} - 1\right)^{-1/\alpha} \\ &= \left(0.05^{-1.3} + 0.05^{-1.3} - 1\right)^{-1/1.3} \\ &= 0.0296\end{aligned}$$

- calculate probability of **one default**

$$\begin{aligned}\mathbb{P}\left(A_1 \leq 0.05, A_2 > 0.05\right) &= \mathbb{P}\left(A_1 > 0.05, A_2 \leq 0.05\right) \\ &= 0.05 - 0.0296 \\ &= 0.0204\end{aligned}$$

Example: Gaussian vs. Clayton copula

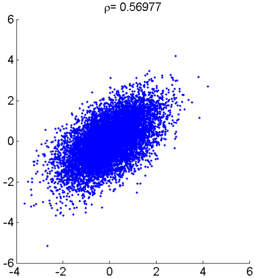
- calculate probability of **no default**

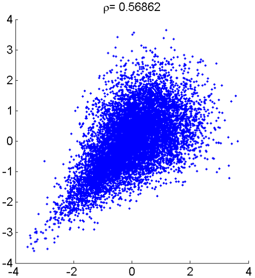
$$\begin{aligned}\mathbb{P}\left(A_1 > 0.05, A_2 > 0.05\right) &= 1 - 2 \cdot \beta - \alpha \\ &= 1 - 2 \cdot 0.0204 - 0.0296 \\ &= 0.9296\end{aligned}$$

		X_1		
		1	0	
X_2	0	0.02	0.93	0.95
	1	0.03	0.02	0.05
		0.05	0.95	

Example: Gaussian vs. Clayton copula

- assuming that underlying **asset** paths follow individual standard normal distributions, both models exhibit **nearly equal correlations**
- while the left model is usually used to derive joint default probabilities in practice, the **right** model exhibits **substantially higher joint default probabilities** (0.3 instead of 0.15)
- moreover, since both models exhibit equal correlations, practitioners might even not be aware of the risk due to possibly wrong model specifications





Example: Gaussian vs. Clayton copula

- **measuring** model differences
  - **default correlations**

$$\varrho^{Gau}_{X_1X_2} = \frac{\alpha - p^2}{p\,(1-p)} = \frac{0.015 - 0.05^2}{0.05 \cdot 0.95} = 0.263$$

$$\varrho^{Clay}_{X_1X_2} = \frac{\alpha - p^2}{p\,(1-p)} = \frac{0.03 - 0.05^2}{0.05 \cdot 0.95} = 0.580$$

  - portfolio defaults

	Gaussian			Clayton		
number defaults	0	1	2	0	1	2
probability	0.915	0.07	0.015	0.930	0.040	0.030
cumulative	0.915	0.985	1	0.930	0.970	1

  - default value-at-risk: $VaR^{Gau}_{0.98} = 1,\ VaR^{Clay}_{0.98} = 2$

Example: Gaussian vs. Clayton copula

- without copulas, would such joint default probabilities ever be imaginable?
- calculating **Gaussian copula** parameter ρ, **leading to equally large joint default** probability of 0.03 with individual default probabilities of 0.05 requires solution of

$$C_{\rho}^{Gau}(0.05, 0.05) \stackrel{!}{=} 0.03$$

- trial and error **approximation** leads to $\rho = 0.88$:

$$C_{0.88}^{Gau}(0.05, 0.05) = 0.0302$$

- practitioners could deem such a high correlation between assets as very unlikely!

Factor model

Goal: determine **asset correlation**

- asset returns are assumed to be a **composite** of influences from individual country and industry **factors** $(X_i)_{1 \leq i \leq n}$ and a firm specific **idiosyncratic component** ϵ, $\epsilon \sim \mathcal{N}(0,1)$, and factors and idiosyncratic component are assumed to be uncorrelated:

$$Cov\left(X_i, \epsilon\right) = 0$$

- for the case of **two factors** we get

$$r = a_1 X_1 + a_2 X_2 + b\epsilon$$

- the associated **asset return variance** is given by

$$\mathbb{V}(r) = a_1^2 \mathbb{V}(X_1) + a_2^2 \mathbb{V}(X_2) + 2 a_1 a_2 Cov(X_1, X_2) + b^2 \mathbb{V}(\epsilon)$$

Variance decomposition

- the asset return **variance** can be **decomposed** into a part arising from common market fluctuations and a part arising from idiosyncratic components
- variance of **non-idiosyncratic** part:

$$\begin{aligned}\mathbb{V}\left(a_1 X_1 + a_2 X_2\right) &= a_1^2 \mathbb{V}\left(X_1\right) + a_2^2 \mathbb{V}\left(X_2\right) + 2 a_1 a_2 Cov\left(X_1, X_2\right) \\ &= a_1^2 \sigma_{X_1}^2 + a_2^2 \sigma_{X_2}^2 + 2 a_1 a_2 \rho_{X_1, X_2} \sigma_{X_1} \sigma_{X_2}\end{aligned}$$

- instead of explicitly specifying coefficient b, it suffices to know **fraction** c **of variance explained** by non-idiosyncratic components in order to calculate overall variance

$$\mathbb{V}\left(r\right) = \frac{\mathbb{V}\left(a_1 X_1 + a_2 X_2\right)}{c}$$

Correlations from factor models

- given factor models of **two assets**

$$r = a_1 X_1 + a_2 X_2 + b\epsilon,$$

$$\mathfrak{r} = \mathfrak{a}_1 X_3 + \mathfrak{a}_2 X_4 + b\varepsilon,$$

asset **covariance** is given by

$$\begin{aligned} Cov\left(r, \mathfrak{r}\right) =& a_1 \mathfrak{a}_1 Cov\left(X_1, X_3\right) + a_1 \mathfrak{a}_2 Cov\left(X_1, X_4\right) \\ & + a_2 \mathfrak{a}_1 Cov\left(X_2, X_3\right) + a_2 \mathfrak{a}_2 Cov\left(X_2, X_4\right), \end{aligned}$$

- hence, **asset correlation** can be calculated according to the standard formula by

$$\rho_{r,\mathfrak{r}} = \frac{Cov\left(r, \mathfrak{r}\right)}{\sigma_r \sigma_{\mathfrak{r}}}$$

Example

The company *ABC* is associated with country-specific risks of countries Germany and Spain. According to the financial analyst in charge, the cash-flows of the company are produced in both countries at a ratio of 3 to 1, and the fraction of the overall variance explained by country specific factors is 0.4. Determine the coefficients of the factor model, when the volatility of the German index is $\sigma_G = 1.4$, the volatility of the Spain index is $\sigma_S = 1.2$ and their correlation is given by $\rho_{G,S} = 0.3$.

- model specification:

$$r = 0.75X_G + 0.25X_S + b\epsilon, \quad \epsilon \sim \mathcal{N}(0, 1)$$

Example

- variance induced by country specific factors

$$\begin{aligned}\mathbb{V}\left(0.75X_G+0.25X_S\right)=&0.75^2\mathbb{V}\left(X_G\right)+0.25^2\mathbb{V}\left(X_S\right)\\&+2\cdot 0.75\cdot 0.25\,Cov\left(X_G,X_S\right)\\=&0.75^2\cdot 1.4^2+0.25^2\cdot 1.2^2\\&+2\cdot 0.75\cdot 0.25\cdot(0.3\cdot 1.4\cdot 1.2)\\=&1.3815\end{aligned}$$

- overall variance:

$$0.4\cdot\mathbb{V}\left(r\right)=1.3815$$

$$\mathbb{V}\left(r\right)=\frac{1.3815}{0.4}=3.4537$$

Example

Remarks:

- overall **variance** in factor model **depends on scaling**:
  - instead of coefficients 0.75 and 0.25 we also could have taken values 3 and 1, or any other multiple, leading to different asset variances
  - covariance between assets depends on scaling, too
- however, through focussing on **correlations**, units of measurement become **normalized**: scaling effects drop out for both assets