Risk management

Functions applied to random variables

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Univariate transformations

Target

- target random variable Y: **function of random variable** with know distribution

$$\begin{aligned} X &\sim F_X \\ Y &= g(X) \\ Y &\sim ? \end{aligned}$$

Example: discrete case

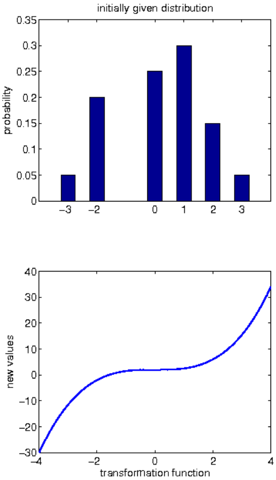
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−3
−2
0
1
2
3
new values
40
30
20
10
0
−10
−20
−30
−4
−2
0
2
4
transformation function

Figure 1:

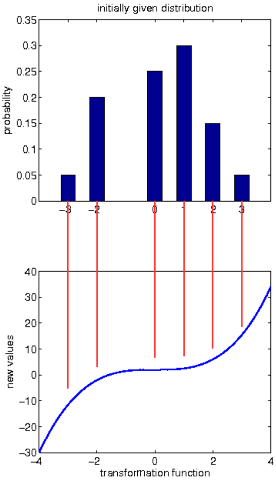
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−3
−2
0
1
2
3
new values
40
30
20
10
0
−10
−20
−30
−4
−2
0
2
4
transformation function

Figure 2:

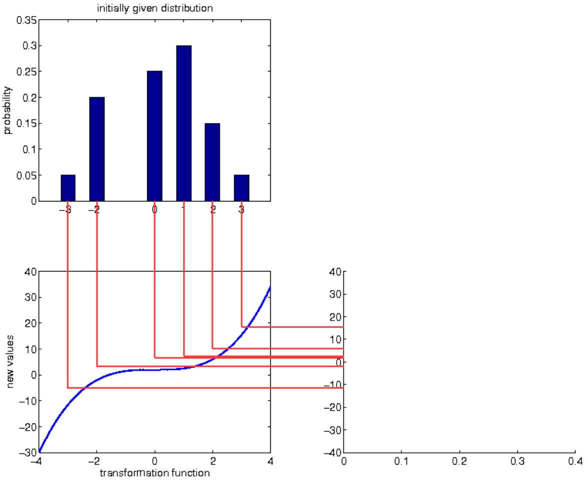
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−3
−2
0
1
2
3
new values
40
30
20
10
0
−10
−20
−30
−4
−2
0
2
4
transformation function
40
30
20
10
0
−10
−20
−30
−40
0
0.1
0.2
0.3
0.4

Figure 3:

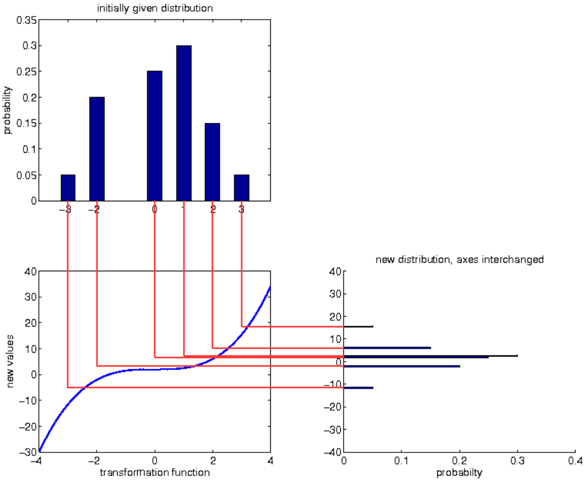
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−3
−2
0
1
2
3
new values
40
30
20
10
0
−10
−20
−30
−4
−2
0
2
4
transformation function
new distribution, axes interchanged
40
30
20
10
0
−10
−20
−30
−40
0
0.1
0.2
0.3
0.4
probabilty

Figure 4:

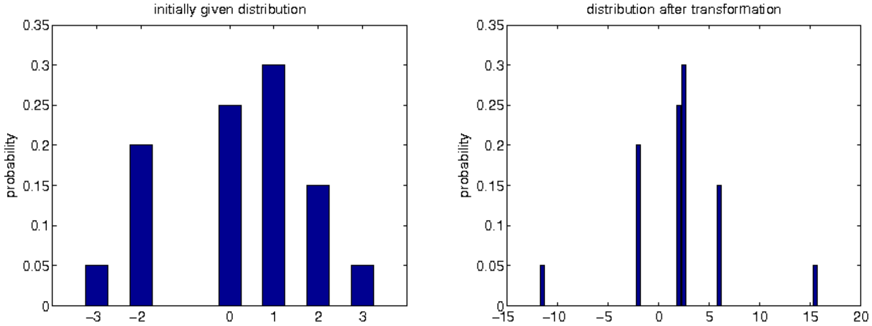
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−3
−2
0
1
2
3
distribution after transformation
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−15
−10
−5
0
5
10
15
20

Figure 5:

Example: call option payoff

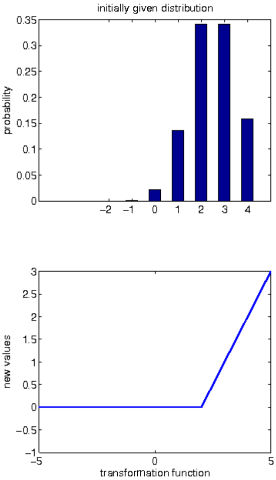
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−2
−1
0
1
2
3
4
new values
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−5
0
5
transformation function

Figure 6:

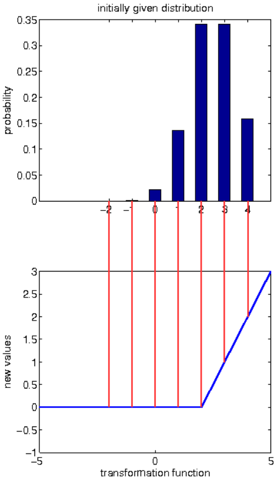
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−2
−1
0
1
2
3
4
new values
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−5
0
5
transformation function

Figure 7:

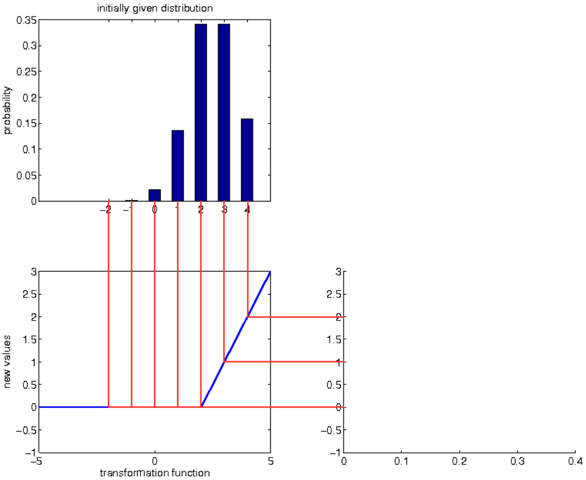
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−2
−1
0
1
2
3
4
new values
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−5
0
5
transformation function
0
0.1
0.2
0.3
0.4

Figure 8:

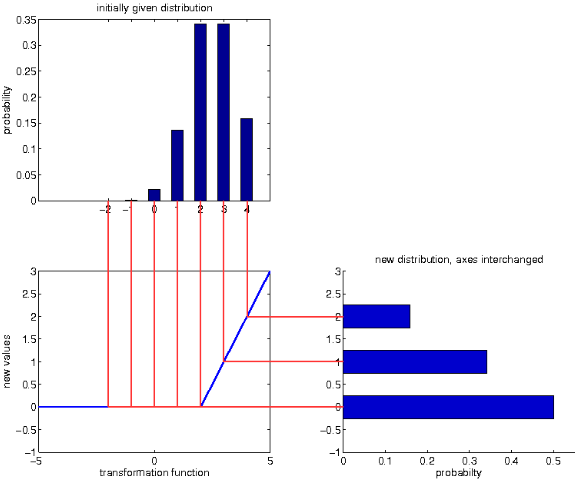
initially given distribution
probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−2
−1
0
1
2
3
4
new values
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−5
0
5
transformation function
new distribution, axes interchanged
0
0.1
0.2
0.3
0.4
0.5
probabilty

Figure 9:

Transformations of continuous random variables

Analytical formula

Transformation theorem

Let X be a random variable with density function $f(x)$, and $g(x)$ be an **invertible bijective** function. Then the density function of the **transformed random variable** $Y = g(X)$ in any point z is given by

$$f_Y(z) = f_X\left(g^{-1}(z)\right) \cdot \left|\left(g^{-1}\right)'(z)\right|.$$

Problems

- **given** that we can calculate **some measure** ϱ_X of the original random variable X, it is **not ensured** that ϱ_Y **can be calculated** for the new random variable Y, too: e.g. if ϱ envolves integration
- what about non-invertible functions?

Example: return distribution

- traditional financial modeling: normally distributed **logarithmic returns**

- for example: **percentage logarithmic returns**

$$R^{\log} := 100 r^{\log}$$

- net returns as function of $R^{\log}$:

$$r = \exp(R^{log}/100) - 1$$

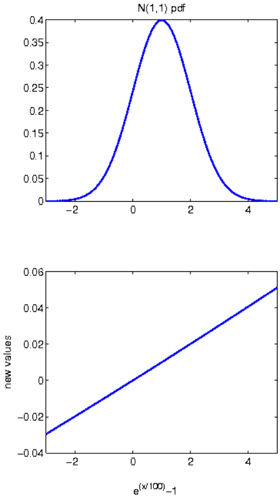
N(1,1) pdf
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−2
0
2
4
0.06
0.04
0.02
0
−0.02
−0.04
new values
−2
0
2
4
e^(x/100)−1

Figure 10:

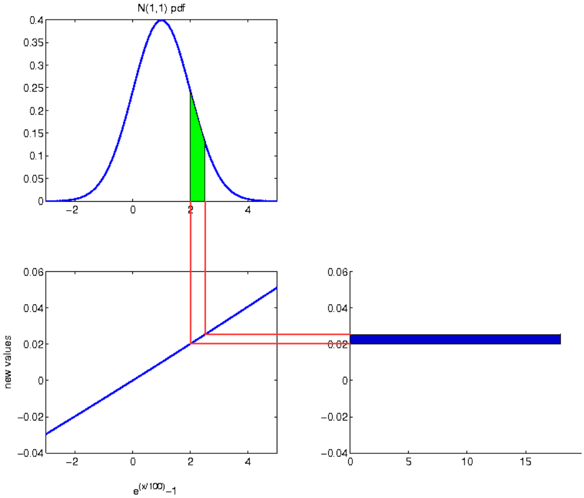
N(1,1) pdf
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−2
0
2
4
0.06
0.04
0.02
0
−0.02
−0.04
new values
$e^{(x/100)}-1$
5
10
15

Figure 11:

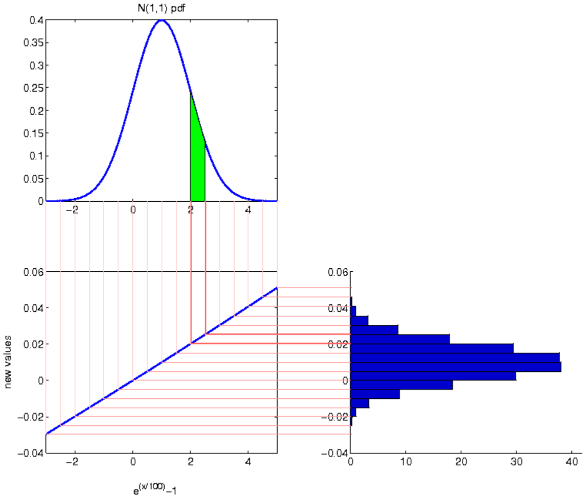
N(1,1) pdf
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−2
0
2
4
0.06
0.04
0.02
0
−0.02
−0.04
new values
$e^{(x/100)}-1$
0
10
20
30
40

Figure 12:

Analytical calculation

- according to the transformation theorem, we get for the distribution of **net returns**:

$$f_r(z) = f_{R^{log}}\left(g^{-1}(z)\right) \cdot \left|\left(g^{-1}\right)'(z)\right|$$
$$g(x) = e^{x/100} - 1$$

$\Rightarrow$ calculating each part

- calculation of g^{-1} :

$$\begin{aligned} x &= e^{y/100} - 1 \Leftrightarrow \\ x + 1 &= e^{y/100} \qquad \Leftrightarrow \\ \log(x+1) &= y/100 \qquad \Leftrightarrow \\ 100 \cdot \log(x+1) &= y \end{aligned}$$

- calculation of $\left(g^{-1}\right)'$:

$$\left(100 \cdot \log(x+1)\right)' = 100 \cdot \frac{1}{x+1}$$

- plugging in leads to:

$$f_r(z) = f_{R^{log}}\left(100 \cdot \log\left(z+1\right)\right) \cdot \left|\frac{100}{z+1}\right|$$

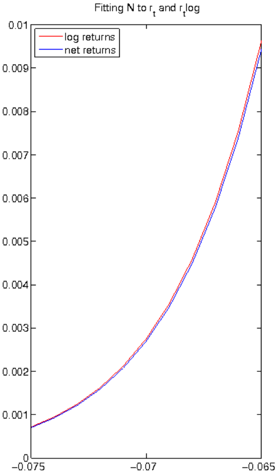
Fitting N to r_t and r_t log
log returns
net returns
0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
−0.075
−0.07
−0.065
Fitting N to r_t and r_t log
log returns
net returns
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.055
0.06
0.065
0.07

Figure 13:

- although only visable under magnification, **there is a difference** between a normal distribution which is directly fitted to net returns and the distribution which arises for net returns by fitting a normal distribution to logarithmic returns
- the resulting distribution from fitting a normal distribution to logarithmic returns assigns **more probability to extreme negative returns** as well as less probability to extreme positive returns

Example: inverse normal distribution

- application of an inverse normal cumulative distribution as transformation function to a uniformly distributed random variable

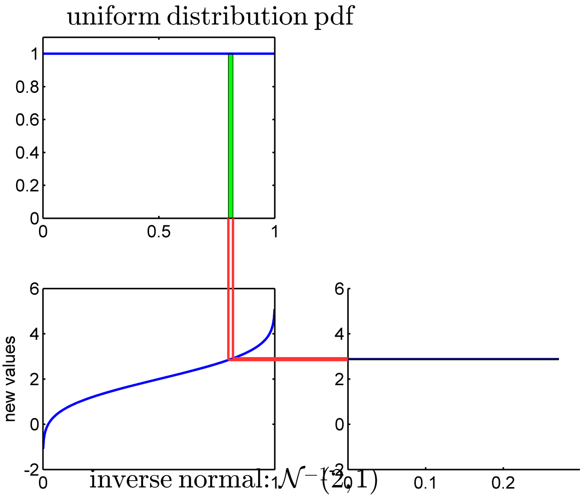
uniform distribution pdf
1
0.8
0.6
0.4
0.2
0
0
0.5
1
new values
6
4
2
0
-2
0
1
inverse normal: $\mathcal{N}^{-1}(2,1)$
6
4
2
0
-2
0
0.1
0.2

Figure 14:

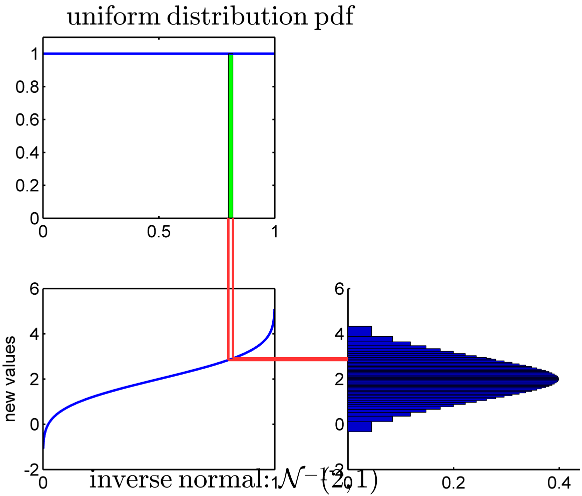
uniform distribution pdf
1
0.8
0.6
0.4
0.2
0
0
0.5
1
6
4
2
0
-2
new values
0
1
inverse normal: $\mathcal{N}^{-1}(2,1)$
6
4
2
0
-2
0.2
0.4

Figure 15:

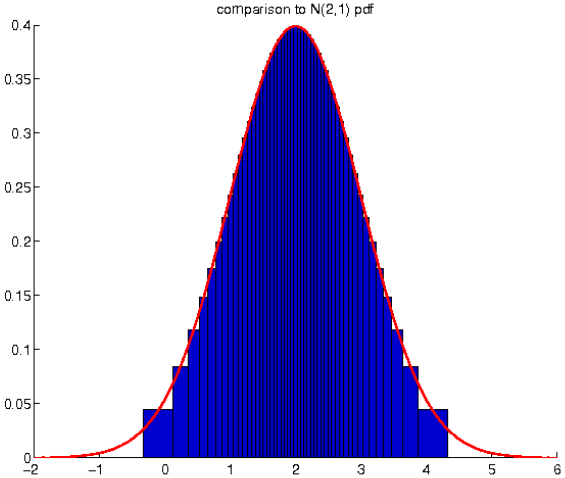

Figure 16:

- the resulting distribution really is the normal distribution
- application of an inverse cdf to a uniformly distributed random variable forms the basis of **Monte Carlo simulation**

Monte Carlo simulation

Let X be a univariate random variable with distribution function F_X. Let F_X^{-1} be the quantile function of F_X, i.e.

$$F_X^{-1}(p) = \inf\{x | F_X(x) \geq p\}, \quad p \in (0,1).$$

Then, we can **simulate random variables** with arbitrary distribution function F_X through:

$$F_X^{-1}(U) \sim F_X, \quad \text{for } U \sim \mathbb{U}[0,1]$$

Proof

Let X be a continuous random variable with cumulative distribution function F_X, and let Y denote the transformed random variable $Y := F_X^{-1}(U)$. Then

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}\left(F_X^{-1}(U) \leq x\right) = \mathbb{P}(U \leq F_X(x)) = F_X(x)$$

so that Y has the same distribution function as X.

- the reverse direction also is important:

Probability integral transformation

If F_X is continuous, then the random variable $F_X(X)$ is standard uniformly distributed, i.e.

$$F_X(X) \sim \mathbb{U}([0,1])$$

and is called **probability integral transform**.

Proof

$$\begin{aligned}\mathbb{P}(F_X(X) \leq u) &= \mathbb{P}(X \leq F_X^{-1}(u)) \\ &= F_X(F_X^{-1}(u)) \\ &= u\end{aligned}$$

$$\Rightarrow \quad \mathbb{P}(F_X(X) \leq u) = \mathbb{P}(U \leq u) \quad U \sim \mathbb{U}([0,1])$$

Linear transformations

- **linear** transformation functions are given by

$$g(x) = ax + b$$

examples of linear functions:

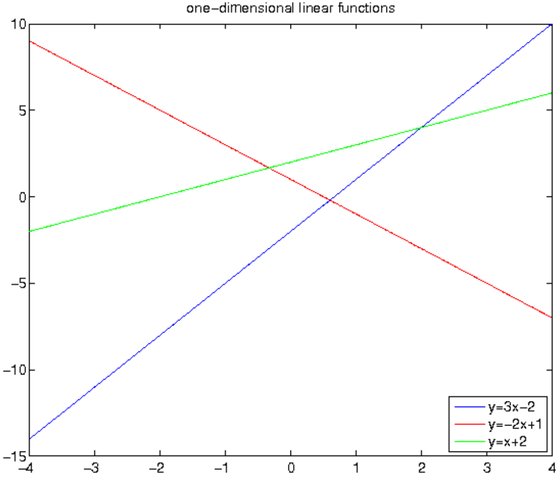

Figure 17:

Analytical solution

- calculate inverse g^{-1} :

$$x = ay + b \Leftrightarrow x - b = ay \Leftrightarrow \frac{x}{a} - \frac{b}{a} = y$$

- calculate derivative $\left(g^{-1}\right)'$:

$$\left(\frac{x}{a} - \frac{b}{a}\right)' = \frac{1}{a}$$

- putting parts together:

$$f_{g(X)}(z) = f_X\left(g^{-1}(z)\right) \cdot \left|\left(g^{-1}\right)'\right| = f_X\left(\frac{z}{a} - \frac{b}{a}\right) \cdot \left|\frac{1}{a}\right|$$

- interpretation: **shifting** b units to the right, **stretching** by factor a

Linear transformation: expectation

- **stretching** and **shifting** also is transferred to the expectation of a **linearly transformed random variable** $Y := aX + b$:

$$\mathbb{E}[Y] = \mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Linear transformation: variance

$$
\begin{aligned}
\mathbb{V}[Y] &= \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right] \\
&= \mathbb{E}\left[(aX + b - \mathbb{E}[aX + b])^2\right] \\
&= \mathbb{E}\left[(aX + b - a\mathbb{E}[X] - b)^2\right] \\
&= \mathbb{E}\left[(a(X - \mathbb{E}[X]) + b - b)^2\right] \\
&= a^2\mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\
&= a^2\mathbb{V}[X]
\end{aligned}
$$

Remarks

- **calculation** of mean and variance of a linearly transformed variable **neither requires** detailed information about the **distribution of** the **original** random variable, **nor** about the distribution of the **transformed** random variable

- knowledge of the respective values of the original distribution is sufficient

- for **non-linear** transformations, such simple formulas do **not exist**

- most situations require **simulation** of the transformed random variable **and subsequent calculation** of the sample value of a given measure